doi:10.3969/j.issn.1005-3026.2017.03.030

求解特定鞍点问题的改进 SOR–Like 方法

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摘 要:鞍点问题广泛出现在众多的工程研究领域,如流体力学、电磁学、最优化问题、最小二乘问题、椭圆偏微分方程问题等.以SOR类方法为基础,结合HS分裂思想,将经典鞍点问题的求解方法推广到特殊鞍点问题的求解上.给出一种具有新型分裂迭代格式的MSOR–Like方法,用以求解一类含有非对称块的鞍点系统,给出了相应的收敛性分析以及最优松弛参数选取方法.数值算例验证了对于不同的预优矩阵,MSOR–Like方法只有收敛速度的分别,没有收敛性能的影响,且在相同计算精度下,该方法解决特殊鞍点问题的迭代效果优于常规方法解决经典鞍点问题.

关 键 词:鞍点问题;迭代法;HS分裂;SOR方法;收敛

中图分类号:O 241 **文献标志码**:A **文章编号**:1005-3026(2017)03-0452-05

Modified SOR-Like Method for Saddle Point Problems

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Abstract: Saddle point problems exist in many engineering research areas such as fluid mechanics, electromagnetism, optimization problems, the least squares problems, elliptic partial differential equations, and etc. Based on SOR-Like methods in combination of the concept of HS splitting, a new iteration splitting improvement method was presented so as to apply the classic saddle point solutions to special saddle point problems. Then, the MSOR-Like method was proposed to handle the above special saddle point system containing asymmetric blocks, and the convergence analysis as well as the selection of optimal relaxation parameters were also given. Finally, a numerical example was given to verify different optimal matrix of the modified SOR method, and it was found that the only difference is in the convergence rate while there is no difference in the convergence effect. Furthermore, under the same calculation accuracy, the modified SOR method for solving the special saddle point problems is better than the conventional methods in solving the classical saddle point problems.

Key words: saddle point problem; iterative method; Hermitian and Skew-Hermitian (HS) splitting; SOR method; convergence

在工程应用的诸多研究中,如涉及流体力学问题、电磁学问题、最优化问题等相关计算时,常常需要求解一类具有特殊结构的鞍点问题[1-5]:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ -\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}=\begin{pmatrix} \boldsymbol{f} \\ -\boldsymbol{g} \end{pmatrix}. \tag{1}$$

其中:$\boldsymbol{A}\in\mathbf{R}^{m\times m}$为非对称的正定矩阵;$\boldsymbol{B}\in\mathbf{R}^{m\times n}$($m\geqslant n$)为列满秩矩阵;$\boldsymbol{x},\boldsymbol{f}\in\mathbf{R}^{m}$;$\boldsymbol{y},\boldsymbol{g}\in\mathbf{R}^{n}$;$\boldsymbol{B}^{\mathrm{T}}$为$\boldsymbol{B}$的转置矩阵;$\boldsymbol{f},\boldsymbol{g}$为已知向量.鞍点系统呈现一定病态,使得一些经典求解方法难以实现,因此如何行之有效地解决这类问题是近年来国内外学者们研究的热点.通常做法是采用定常迭代与非定常迭代双线并行,结合相应的预条件处理技术,其中包括Uzawa类方法[6-9],SOR类方法[10-18],HSS类方法[19-21],Krylov子空间类方法[21],以及相应的预处理方法.

收稿日期:2015-10-21
基金项目:国家自然科学基金资助项目(11371081).
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1　MSOR－Like 方法的迭代格式和收敛性分析

基于 HS 分裂，令 $\boldsymbol{A}=\boldsymbol{H}+\boldsymbol{S}$，其中 $\boldsymbol{H}=\frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})$，$\boldsymbol{S}=\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^{\mathrm{T}})$. 由于 $\boldsymbol{A}$ 为正定矩阵，则 $\boldsymbol{H}$ 也是正定的. 作分裂 $\boldsymbol{M}=\boldsymbol{D}-\boldsymbol{L}-\boldsymbol{U}$，其中 $\boldsymbol{D}=\begin{pmatrix}\boldsymbol{H} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Q}\end{pmatrix}$，$\boldsymbol{L}=\begin{pmatrix}-\boldsymbol{S} & \boldsymbol{0}\\ \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{0}\end{pmatrix}$，$\boldsymbol{U}=\begin{pmatrix}\boldsymbol{0} & -\boldsymbol{B}\\ \boldsymbol{0} & \boldsymbol{Q}\end{pmatrix}$. 这里 $\boldsymbol{Q}\in\mathbf{R}^{n\times n}$ 为对称正定矩阵，若设 $\omega>0$ 为松弛参数，建立迭代格式：

$$\begin{pmatrix}x^{(k+1)}\\ y^{(k+1)}\end{pmatrix}=\boldsymbol{M}_\omega\begin{pmatrix}x^{(k)}\\ y^{(k)}\end{pmatrix}+\boldsymbol{N}_\omega\begin{pmatrix}\boldsymbol{f}\\ -\boldsymbol{g}\end{pmatrix}.$$

其中：

$$\left.\begin{aligned}\boldsymbol{M}_\omega&=\begin{pmatrix}(1-\omega)(\boldsymbol{H}+\omega\boldsymbol{S})^{-1}\boldsymbol{H} & -\omega(\boldsymbol{H}+\omega\boldsymbol{S})^{-1}\boldsymbol{B};\\ \omega(1-\omega)\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{H}+\omega\boldsymbol{S})^{-1}\boldsymbol{H} & \boldsymbol{I}-\omega^2\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{H}+\omega\boldsymbol{S})^{-1}\boldsymbol{B}\end{pmatrix};\\ \boldsymbol{N}_\omega&=\begin{pmatrix}\omega(\boldsymbol{H}+\omega\boldsymbol{S})^{-1} & \boldsymbol{0}\\ \omega^2\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{H}+\omega\boldsymbol{S})^{-1} & \omega\boldsymbol{Q}^{-1}\end{pmatrix}.\end{aligned}\right\}\tag{2}$$

整理得如下迭代算法：

$$\left.\begin{aligned}x^{(k+1)}&=(1-\omega)(\boldsymbol{H}+\omega\boldsymbol{S})^{-1}\boldsymbol{H}x^{(k)}+\omega(\boldsymbol{H}+\omega\boldsymbol{S})^{-1}(\boldsymbol{f}-\boldsymbol{B}y^{(k)}),\\ y^{(k+1)}&=y^{(k)}+\omega\boldsymbol{Q}^{-1}(\boldsymbol{B}^{\mathrm{T}}x^{(k+1)}-\boldsymbol{g}).\end{aligned}\right\}\tag{3}$$

将式(3)称之为求解鞍点问题的 MSOR－Like 方法.

若令 λ 为 MSOR－Like 方法迭代矩阵 $\boldsymbol{M}_\omega$ 的特征值，$(\boldsymbol{x}\quad\boldsymbol{y})^{\mathrm{T}}\in\mathbf{R}^{m+n}$ 为相应特征向量，则有

$$\left.\begin{aligned}&[(1-\omega-\lambda)\boldsymbol{H}-\lambda\omega\boldsymbol{S}]\boldsymbol{x}-\omega\boldsymbol{B}\boldsymbol{y}=0,\\ &\lambda\omega\boldsymbol{B}^{\mathrm{T}}\boldsymbol{x}-(\lambda-1)\boldsymbol{Q}\boldsymbol{y}=0.\end{aligned}\right\}\tag{4}$$

引理 1　设 $\boldsymbol{A}\in\mathbf{R}^{m\times m}$ 为非对称正定矩阵，$\boldsymbol{H}=\frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})$ 为对称正定矩阵，$\boldsymbol{S}=\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^{\mathrm{T}})$ 为反对称矩阵，$\boldsymbol{B}\in\mathbf{R}^{m\times n}(m\geqslant n)$ 为列满秩矩阵，$\boldsymbol{M}_\omega$ 如式(2)，λ 为 $\boldsymbol{M}_\omega$ 的任一特征值，则 $\lambda\neq1$.

证明　(反证法)若 $\lambda=1$，由式(4)可得 $\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ -\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}=\boldsymbol{0}$. 由于系数矩阵是非奇异阵，则有 $(\boldsymbol{x}\quad\boldsymbol{y})^{\mathrm{T}}=0$，这与其为迭代矩阵 $\boldsymbol{M}_\omega$ 的特征向量矛盾，故 $\lambda\neq1$.

引理 2　在引理 1 条件下，设 λ 为迭代矩阵 $\boldsymbol{M}_\omega$ 的任一特征值，$(\boldsymbol{x}\quad\boldsymbol{y})^{\mathrm{T}}\in\mathbf{R}^{m+n}$ 为 $\boldsymbol{M}_\omega$ 的相应特征向量，必有 $\boldsymbol{x}\neq\boldsymbol{0}$. 若 $\boldsymbol{y}=\boldsymbol{0}$，则 $|\lambda|<1$.

证明　①(反证法)若 $\boldsymbol{x}=\boldsymbol{0}$，由式(4)可得 $\boldsymbol{B}\boldsymbol{y}=\boldsymbol{0}$. 由于 $\boldsymbol{B}$ 为列满秩矩阵，因而有 $\boldsymbol{y}=\boldsymbol{0}$，与 $(\boldsymbol{x}\quad\boldsymbol{y})^{\mathrm{T}}$ 为迭代矩阵 $\boldsymbol{M}_\omega$ 的特征向量矛盾，故 $\boldsymbol{x}\neq\boldsymbol{0}$. ② 若 $\boldsymbol{x}\neq\boldsymbol{0}$，$\boldsymbol{y}=\boldsymbol{0}$，由式(4)可得 $(1-\omega-\lambda)\boldsymbol{H}\boldsymbol{x}-\lambda\omega\boldsymbol{S}\boldsymbol{x}=0$. 用 $\frac{\boldsymbol{x}^{\mathrm{T}}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$ 左乘等式两边，则有 $(1-\omega-\lambda)\times\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}-\lambda\omega\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}=0$. 设 $a=\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$，$b=\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$，则有 $(1-\omega-\lambda)a-\lambda\omega b=0$. 由于 $\boldsymbol{H}$ 是对称正定矩阵，则 $a>0$；$\boldsymbol{S}$ 是反对称矩阵，则 $b=0$. 因而，有 $\lambda=1-\omega$. 由于松弛参数需满足 $0<\omega<2$，则有 $|\lambda|=|1-\omega|<1$.

引理 3[1]　实一元二次方程 $\lambda^2-b\lambda+c=0$ 两根之模均小于 1，当且仅当方程系数满足 $|c|<1$，且 $|b|<1+c$.

定理 1　设 $\boldsymbol{A}\in\mathbf{R}^{m\times m}$ 为非对称的正定矩阵，$\boldsymbol{H}=\frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})$ 为对称正定矩阵，$\boldsymbol{S}=\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^{\mathrm{T}})$ 为反对称矩阵，$\boldsymbol{B}\in\mathbf{R}^{m\times n}(m\geqslant n)$ 为列满秩矩阵，$\boldsymbol{Q}\in\mathbf{R}^{n\times n}$ 为适当的对称正定矩阵，则 MSOR－Like 方法迭代格式(3)收敛，当且仅当松弛参数满足 $0<\omega<\frac{-a+\sqrt{a^2+4ac}}{c}$，其中 $a=\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}>0$，$c=\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}>0$.

证明　设 λ 为迭代矩阵 $\boldsymbol{M}_\omega$ 的任一特征值，$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\in\mathbf{R}^{m+n}$ 为 $\boldsymbol{M}_\omega$ 的相应特征向量. 由于 $\boldsymbol{Q}$ 的非奇异性，由引理 1 知 $\lambda\neq1$，将式(4)等价变为

$$\begin{cases}[(1-\omega-\lambda)\boldsymbol{H}-\lambda\omega\boldsymbol{S}]\boldsymbol{x}-\omega\boldsymbol{B}\boldsymbol{y}=0,\\ \boldsymbol{y}=\dfrac{\lambda}{\lambda-1}\omega\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{x}.\end{cases}$$

整理并同时左乘 $\frac{\boldsymbol{x}^{\mathrm{T}}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$ 到等式两边，则有 $(1-\omega-\lambda)\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}-\lambda\omega\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}-\frac{\lambda}{\lambda-1}\omega^2\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}=0$. 设 $a=\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$，$b=\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$，$c=\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$，则有 $(1-\omega-\lambda)a-\lambda\omega b-\frac{\lambda}{\lambda-1}\omega^2c=0$. 由于 $\boldsymbol{H}$ 和 $\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}$ 是对称正定矩阵，因而 $a>0$，$c>0$. 而 $\boldsymbol{S}$ 是反对称矩阵，则 $b=0$，于是有 λ^2+

$\left(\frac{c}{a}\omega^2+\omega-2\right)\lambda+(1-\omega)=0$. 由引理 3 知,

$$|\lambda|<1 \text{当且仅当} \begin{cases} 0<\omega<2, \\ 0<\omega<\frac{-a+\sqrt{a^2+4ac}}{c}. \end{cases} \text{由于}$$

$a>0$,则 $\frac{-a+\sqrt{a^2+4ac}}{c}=\frac{4}{1+\sqrt{1+4\frac{c}{a}}}<2$. 因而,若使 MSOR - Like 方法迭代格式收敛,松弛参数 ω 的范围需满足 $0<\omega<\frac{-a+\sqrt{a^2+4ac}}{c}$.

推论 1 在定理 1 条件下,若 MSOR - Like 方法迭代格式收敛,则松弛参数范围可进一步化简为 $0<\omega<\frac{4}{1+\sqrt{1+4\frac{\rho(BQ^{-1}B^{\mathrm T})}{\lambda_{\min}(H)}}}$,其中 $a=\frac{x^{\mathrm T}Hx}{x^{\mathrm T}x}>0, c=\frac{x^{\mathrm T}BQ^{-1}B^{\mathrm T}x}{x^{\mathrm T}x}>0$.

证明 由于 H 和 $BQ^{-1}B^{\mathrm T}$ 是对称正定矩阵,则 $a=\frac{x^{\mathrm T}Hx}{x^{\mathrm T}x}, c=\frac{x^{\mathrm T}BQ^{-1}B^{\mathrm T}x}{x^{\mathrm T}x}$分别为 H 和 $BQ^{-1}B^{\mathrm T}$ 的 *Rayleigh* 商. 因而, $\frac{-a+\sqrt{a^2+4ac}}{c}=$

$$\frac{4}{1+\sqrt{1+4\frac{c}{a}}}=\frac{4}{1+\sqrt{1+4\frac{\rho(BQ^{-1}B^{\mathrm T})}{\lambda_{\min}(H)}}}.$$

2 数值实验

考虑模型 Stokes 问题,有对流扩散方程:

$$\begin{cases} -\nu\Delta u+\nabla p=f, & \text{in } \Omega; \\ \nabla u=g, & \text{in } \Omega \\ u=0, & \text{on } \partial\Omega; \\ \int_\Omega p(x)\mathrm{d}x=0. \end{cases}$$

其中:Ω 为 $(0,1)\times(0,1)\subset\mathbf{R}^2$; 方程满足 Dirichlet 边界条件; Δ 为 Laplace 算子; u 为表示速度的向量函数; p 为表示压力的数值函数.

将上述问题用迎风差分格式离散化处理,可得形式如下的鞍点问题:

$$\begin{pmatrix} A & B \\ -B^{\mathrm T} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} f \\ -g \end{pmatrix}.$$

其中: $A=\begin{pmatrix} I\otimes T+T\otimes I & 0 \\ 0 & I\otimes T+T\otimes I \end{pmatrix}_{m\times m}$;

$B=\begin{pmatrix} I\otimes F \\ F\otimes I \end{pmatrix}_{m\times n}$.

$T=\frac{1}{h^2}\mathrm{tridiag}(-1,2,-1)\in\mathbf{R}^{p\times p}$, $F=\frac{1}{h}\mathrm{tridiag}(-1,1,0)\in\mathbf{R}^{p\times p}$, $\otimes$ 为 Kronecker 量积符号, $h=\frac{1}{p+1}$为离散网格值,且 $m=2p^2, n=p^2$. 将模型中的 T 作适当修改,得到 $\tilde{T}=\frac{1}{h^2}\mathrm{tridiag}(-1.5,2,-0.5)\in\mathbf{R}^{p\times p}$. 于是问题模型重新定义为

$$A=\begin{pmatrix} I\otimes\tilde T+\tilde T\otimes I & 0 \\ 0 & I\otimes\tilde T+\tilde T\otimes I \end{pmatrix}_{m\times m},$$

$$B=\begin{pmatrix} I\otimes F \\ F\otimes I \end{pmatrix}_{m\times n}.$$

令 $A=H+S$,其中 $H=\frac{1}{2}(A+A^{\mathrm T})$, $S=\frac{1}{2}(A-A^{\mathrm T})$,有 $\tilde T=T_1+T_2$,其中 $T_1=\frac{1}{2}(\tilde T+\tilde T^{\mathrm T})$, $T_2=\frac{1}{2}(\tilde T-\tilde T^{\mathrm T})$.

取 $h=\frac{1}{p+1}$,则 $m=2p^2, n=p^2$. 取零向量为初始向量,其精确解为 $(x^{\mathrm T},y^{\mathrm T})^{\mathrm T}=(1,1,\cdots,1)^{\mathrm T}$. 设 IT 为迭代步数,CPU 为上机运算满足收敛要求所需时间,使用 MATLAB 编程语言实现,当 ERR $<10^{-6}$时计算中止,其中 ERR 表示迭代误差,定义如下:

$$\mathrm{ERR}=\frac{\sqrt{\|x^{(k)}-x^*\|_2^2+\|y^{(k)}-y^*\|_2^2}}{\sqrt{\|x^{(0)}-x^*\|_2^2+\|y^{(0)}-y^*\|_2^2}}.$$

Case 1 $Q=\frac{1}{20}I$,

$$\begin{cases} x^{(k+1)}=(1-\omega)(H+\omega S)^{-1}Hx^{(k)}+ \\ \quad\omega(H+\omega S)^{-1}(f-By^{(k)}), \\ y^{(k+1)}=y^{(k)}+20\omega(B^{\mathrm T}x^{(k+1)}-g). \end{cases}$$

表 1 $h=1/9\,(p=8, m=128, n=64)$时,$\omega_{\mathrm{opt}}$附近区域收敛情况(最优松弛参数 $\omega_{\mathrm{opt}}=0.25$)

Table 1 Converging situation around optimal omega with $h=1/9\,(p=8, m=128, n=64)\,(\omega_{\mathrm{opt}}=0.25)$

ω	IT	CPU	ERR
0.10	276	2.184 0	9.501 0e - 07
0.15	193	1.638 0	8.882 2e - 07
0.20	155	1.216 8	8.849 8e - 07
0.25	141	1.060 8	9.162 5e - 07
0.30	158	1.263 6	8.837 1e - 07

Case 2　$Q = B^{\mathrm{T}}A^{-1}B$, $\begin{cases} x^{(k+1)} = (1-\omega)(H+\omega S)^{-1}Hx^{(k)} + \omega(H+\omega S)^{-1}(f - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \omega(B^{\mathrm{T}}A^{-1}B)^{-1}(B^{\mathrm{T}}x^{(k+1)} - g). \end{cases}$

表 2　取 $h=1/9$ ($p=8, m=128, n=64$) 时, ω_{opt} 附近区域收敛情况 (最优松弛参数 $\omega_{opt}=1.00$)
Table 2　Converging situation around optimal omega with $h=1/9$ ($p=8, m=128, n=64$) ($\omega_{opt}=1.00$)

ω	IT	CPU	ERR
0.98	9	0.093 6	5.698 5e－08
0.99	7	0.062 4	8.099 8e－07
1.00	2	0.031 2	9.488 1e－16
1.01	8	0.062 4	1.952 4e－07
1.02	9	0.093 6	6.221 3e－07

Case 3　$Q = B^{\mathrm{T}}H^{-1}B$, $\begin{cases} x^{(k+1)} = (1-\omega)(H+\omega S)^{-1}Hx^{(k)} + \omega(H+\omega S)^{-1}(f - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \omega(B^{\mathrm{T}}H^{-1}B)^{-1}(B^{\mathrm{T}}x^{(k+1)} - g). \end{cases}$

表 3　取 $h=1/9$ ($p=8, m=128, n=64$) 时, ω_{opt} 附近区域收敛情况 (最优松弛参数 $\omega_{opt}=1.15$)
Table 3　Converging situation around optimal omega with $h=1/9$ ($p=8, m=128, n=64$) ($\omega_{opt}=1.15$)

ω	IT	CPU	ERR
1.00	51	0.468 0	9.946 3e－07
1.05	50	0.405 6	8.140 7e－07
1.10	47	0.327 6	9.673 2e－07
1.15	47	0.421 2	6.614 1e－07
1.20	88	0.655 2	9.756 8e－07

数值实验结果表 1 ~ 表 3 表明,对于不同的预优矩阵 Q,MSOR－Like 方法只有收敛速度的区别,没有收敛性的影响.因而可以得出,在解决非对称鞍点问题上,该方法是切实有效的.

3　结　　语

本文以 SOR－Like 方法为基础,融合 HS 分裂思想,将经典鞍点问题的求解方法推广到特殊鞍点问题的求解,给出了一种具有新型分裂迭代格式的 MSOR－Like 方法,理论分析和数值算例验证了 MSOR－Like 方法在解决非对称鞍点问题时的有效性.

参考文献:

[1] Benzi M, Golub G H, Liesen J. Numerical solution of saddle point problems[J]. *Acta Numerica*, 2005, 14(1): 130－137.

[2] Duff I S, Gould N I M, Reid J K, et al. The factorization of sparse symmetric indefinite matrices[J]. *IMA Journal of Numerical Analysis*, 1991, 11(2): 181－204.

[3] Varga R S. Matrix iterative analysis[M]. Englewood Cliffs: Prentice-Hall, 1962.

[4] Hadjidimos A. Accelerated over relaxation method[J]. *Mathematics of Computation*, 1978, 32(141): 149－157.

[5] Arrow K, Hurwicz L, Uzawa H. Studies in nonlinear programming[M]. Stanford: Stanford University Press, 1958: 5.

[6] Elman H C, Golub G H. Inexact and preconditioned Uzawa algorithms for saddle point problems[J]. *SIAM Journal on Numerical Analysis*, 1994, 31(6): 1645－1661.

[7] Bramble J H, Pasciak J E, Vassilev A T. Analysis of the inexact Uzawa algorithm for saddle point problems[J]. *SIAM Journal on Numerical Analysis*, 1997, 34(3): 1072－1092.

[8] Bai Z Z, Wang Z Q. On parameterized inexact Uzawa methods for generalized saddle point problems[J]. *Linear Algebra and Its Applications*, 2008, 428(11): 2900－2932.

[9] Zhang J, Shang J. A class of Uzawa-SOR methods for saddle point problems[J]. *Applied Mathematics and Computation*, 2010, 216(7): 2163－2168.

[10] Yang A L, Wu Y J. The Uzawa-HSS method for saddle point problems[J]. *Applied Mathematics Letters*, 2014, 38: 38－42.

[11] Yun J H. Variants of the Uzawa method for saddle point problem[J]. *Computers & Mathematics with Applications*, 2013, 65(7): 1037－1046.

[12] Young D M. Iterative solution for large systems[M]. New York: Academic Press, 1971.

[13] Golub G H, Wu X, Yuan J Y. SOR-Like methods for augmented systems[J]. *BIT Numerical Mathematics*, 2001, 41(1): 71－85.

[14] Bai Z Z, Parlett B N, Wang Z Q. On generalized successive overrelaxation methods for augmented linear systems[J]. *Numerische Mathematik*, 2005, 102(1): 1－38.

[15] Li C, Li Z, Nie Y Y, et al. Generalized AOR method for the augmented system [J]. *International Journal of Computer Mathematics*, 2004, 81(4): 495－504.

[16] Darvishi M T, Hessari P. Symmetric SOR method for augmented systems [J]. *Applied Mathematics and Computation*, 2006, 183(1): 409－415.

[17] Zhang G F, Lu Q. On generalized symmetric SOR method for augmented systems [J]. *Journal of Computational and Applied Mathematics*, 2008, 219(1): 51－58.

[18] Bai Z Z, Golub G H, Ng M K. Hermitian and Skew-Hermitian splitting methods for non-Hermitian positive definite linear systems[J]. *SIAM Journal on Matrix Analysis and Applications*, 2003, 24(3): 603－626.

[19] Benzi M, Gander M J, Golub G H. Optimization of the Hermitian and Skew-Hermitian splitting iteration for saddle-point problems [J]. *BIT Numerical Mathematics*, 2003, 43(5): 881－900.

[20] Bai Z Z, Golub G H, Pan J Y. Preconditioned Hermitian and Skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems [J]. *Numerische Mathematik*, 2004, 98(1): 1－32.

[21] Bai Z Z, Golub G H. Accelerated Hermitian and Skew-Hermitian splitting iteration methods for saddle-point problems[J]. *IMA Journal of Numerical Analysis*, 2007, 27(1): 1－23.

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参考文献:

[1] Groth M, Hennig-Thurau T. Customer reactions to emotional labor: the roles of employee acting strategies and customer detection accuracy [J]. *Academy of Management Journal*, 2009, 52(5): 958－974.

[2] Jung H S, Yoon H H. Antecedents and consequences of employees' job stress in a foodservice industry: focused on emotional labor and turnover intent[J]. *International Journal of Hospitality Management*, 2014, 38(1): 84－88.

[3] Maneotis S M, Grandey A A, Krauss A D. Understanding the "why" as well as the "how": service performance is a function of prosocial motives and emotional labor [J]. *Human Performance*, 2014, 27(1): 80－97.

[4] Kammeyer-Mueller J D, Rubenstein A L, Long D M, et al. A meta-analytic structural model of dispositonal affectivity and emotional labor[J]. *Personnel Psychology*, 2013, 66(1): 47－90.

[5] 杨勇, 马钦海, 曾繁强, 等. 组织公平感与情绪劳动策略关系的实证研究[J]. 工业工程与管理, 2013, 18(4): 37－44.
(Yang Yong, Ma Qin-hai, Zeng Fan-qiang, et al. Empirical research on the relationship between organizational justice and emotional labor [J]. *Industrial Engineering and Management*, 2013, 18(4): 37－44.)

[6] Prati L M, Liu Y, Perrewé P L, et al. Emotional intelligence as moderator of the surface acting strain relationship [J]. *Journal of Leadership & Organizational Studies*, 2009, 15(4): 368－379.

[7] Duke A B, Goodman J M, Treadway D C, et al. Perceived organizational support as a moderator of emotional labor/outcomes relationships [J]. *Journal of Applied Social Psychology*, 2009, 39(5): 1013－1034.

[8] Schnelder B, White S S, Paul M C. Linking service climate and customer perceptions of service quality: test of a causal model [J]. *Journal of Applied Psychology*, 1998, 83(2): 150－163.

[9] 杨勇, 唐帅, 马钦海, 等. 情绪智力与组织公民行为: 情绪劳动的中介作用[J]. 东北大学学报(自然科学版), 2013, 34(5): 753－756.
(Yang Yong, Tang Shuai, Ma Qin-hai, et al. Emotional intelligence and organizational citizenship behavior: the mediating role of emotional labor [J]. *Journal of Northeastern University (Natural Science)*, 2013, 34(5): 753－756.)

[10] Farh J L, Earley P C, Lin S C. Impetus for action: a cultural analysis of justice and organizational citizenship behavior in Chinese society [J]. *Administrative Science Quarterly*, 1997, 42(3): 421－444.

[11] Hobfoll S E. Stress, culture, and community: the psychology and philosophy of stress [M]. New York: Plenum Press, 1998.

[12] Walumbwa F O, Suzanne J P, Avolio B J, et al. An investigation of the relationships among leader and follower psychological capital, service climate and job performance [J]. *Personnel Psychology*, 2010, 63(4): 937－963.