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带流体动力学阻尼的IBq方程的精确解

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摘　　要: 对带流体动力学阻尼的IBq方程进行了研究,发现虽然对Bq方程精确解的研究很多,但对IBq方程解的研究结果却很少. 介绍了求解非线性演化方程的Tanh法与扩展Tanh函数法,使用符号计算软件Maple和Tanh函数法获得带流体动力学阻尼的IBq方程的大量双曲函数精确解,主要为扭结和反扭结孤立子解. 对精确解中未知参数进行赋值,图解表示了部分精确解,这对于数值解的准确性和稳定性的核对是有用的. 获得的结果证实该方法用于分析求解数学物理中各种非线性偏微分方程是有效的.

关 键 词: Tanh函数法; 扩展Tanh函数法; 双曲函数精确解; IBq方程; 流体动力学阻尼
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Exact Solutions for IBq Equation with Fluid Dynamic Damping

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Abstract: The IBq equation with fluid dynamic damping was studied. Many studies of exact solutions for Bq equation were found, but the study results of the IBq equations were very few. The standard Tanh method and the extended Tanh method were introduced to solve nonlinear evolution equation, and the standard Tanh method and symbolic computation system Maple were used to obtain a large number of exact hyperbolic function solutions of IBq equation with fluid dynamic damping, mainly for the kink and the antikink soliton solutions. Assignment of exact solutions was done for the unknown parameters, and figures showed some exact solutions, which were useful for verifying the accuracy and stability of numerical solution. The obtained results confirm that the proposed methods are efficient techniques for analytic treatment of a wide variety of nonlinear partial differential equations in mathematical physics.

Key words: Tanh method; extended Tanh method; exact hyperbolic function solutions; IBq equation; fluid dynamic damping

非线性演化方程被广泛用于描述许多重要的现象和动态过程,比如流体力学、等离子体、生物学、光纤和其他工程领域. 理论物理和非线性科学的进步使得可以构造这些非线性方程的精确行波解,借助于数学符号软件Maple或者Mathematica可以直接寻找这些非线性方程的精确解. 近年来,许多对非线性物理现象感兴趣的学者研究了非线性演化方程的精确解的解决方案,并提出许多有效的方法,例如,逆散射法[1]、Tanh函数法[2]、sine-cosine方法[3]、扩展Tanh函数法[4-6]、齐次平衡法[7]、F-expansion法[8]、首次积分法[9]及(G'/G)-函数展开法[5,10]等.

Boussinesq[11]研究长波在浅水波表面传播的问题时,首次导出Boussinesq方程,简称Bq方程. 它的行波解被Wazwaz[12]用Tanh函数法求得. 如果Bq中4阶导数项的系数$\delta>0$,则是线性稳定的,用于描述微小的非线性弹性梁的横向振动[13],被称为'好的'Bq方程,它的行波解被Mohyud等[14]用Exp-function法求得. 当$\delta<0$时,由于它的线性不稳定性,被称为'坏的'Bq方

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程[15]，一个 2 维的坏的 Bq 方程被提出用来描述表面重力波的传播，特别是斜向波的正面碰撞[16]，它的行波解被 Forozani 等[17]用扩展 Tanh 函数法求得. Makhankov[18]从等离子体的流体动力学方程组中获得 IBq 方程，该方程不仅可以用来近似描述长波在浅水水波中传播，还可用于描述非谐单原子和双原子链的动力学与热力学特性[19]. 在真实的过程中，内部摩擦(流体类型的摩擦)同样扮演着重要的角色，它产生于系统内部的不可逆过程中，内部摩擦产生耗散，耗散函数依赖于相对位移的时间导数，因此，有必要探究带流体阻尼(耗散项)的 IBq 方程[17]. 文献[20－21]都有涉及 IBq 方程柯西问题解存在性的研究，但未见文章给出 IBq 方程的精确行波解. 本文将使用标准 Tanh 和扩展 Tanh 法，结合 Maple 符号计算软件分别得到它的精确行波解.

1　Tanh 函数法与扩展 Tanh 函数法概述

1）首先，考虑一个一般形式的非线性偏微分方程：

$$P(u,u_t,u_x,u_{tt},u_{xt},u_{xx},\cdots)=0\ . \qquad (1)$$

式中：$u(x,t)$表示未知函数；下标 t 和 x 表示 $u(x,t)$ 对 t 和 x 的偏导数. 为了找方程(1)的行波解，作行波变换：

$$\xi=x-ct,u(x,t)=U(\xi)\ . \qquad (2)$$

式中，c 为参数. 结合下面的一些变化：

$$\frac{\partial}{\partial t}=-c\frac{\partial}{\partial\xi},\frac{\partial}{\partial x}=\frac{\partial}{\partial\xi},\frac{\partial^2}{\partial x^2}=\frac{\partial^2}{\partial\xi^2},\frac{\partial^2}{\partial x\partial t}=-c\frac{\partial^2}{\partial\xi^2}\cdots$$

把方程(1) 变换为非线性常微分方程：

$$O(U,U',U'',U''',\cdots)=0\ . \qquad (3)$$

2）如果得到的常微分方程每一项都含有 ξ 的导数，则可把这个常微分方程先关于 ξ 积分，令积分常数为零得到一个更为简单的方程.

3）假设一个新的独立变量 $y(\xi)=\tanh(k\xi)$，会引出如下的导数变换式：

$$\frac{\mathrm{d}}{\mathrm{d}(\xi)}=k(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}=k^2(1-y^2)\left[-2y\frac{\mathrm{d}}{\mathrm{d}y}+(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y^2}\right],$$

$$\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}=k^3(1-y^2)\left[(6y^2-2)\frac{\mathrm{d}}{\mathrm{d}y}-6y(1-y^2)\frac{\mathrm{d}^2}{\mathrm{d}y^2}+(1-y^2)\frac{\mathrm{d}^3}{\mathrm{d}y^3}\right]\cdots$$

则标准 Tanh 函数展开法解可拟设为一个关于 $y(\xi)$的有限形式：

$$U(\xi)=S(y)=\sum_{i=0}^{m}a_iy^i. \qquad (4)$$

扩展 Tanh 函数展开法解的形式可拟设为

$$u(x,t)=S(y)=\sum_{i=0}^{m}a_iy^i+\sum_{i=1}^{m}b_iy^{-i}. \qquad (5)$$

4）参数 m 一般都为正整数，为了确定 m 的数值，平衡方程(3)中的线性最高阶和最高阶非线性项的幂次. 当 m 被确定后，收集方程(3)关于 $y(\xi)$的系数，令这些系数方程都为零，当使用方程(4)形式的拟设解时，得到关于 $a_i(i=0,1,\cdots,m)$和 k,c 的方程组；使用方程(5)形式的拟设解时，得到关于 $a_i(i=0,1,\cdots,m)$，$b_i(i=1,2,\cdots,m)$和 k,c 的方程组. 使用 Maple 求解，把求解结果代入方程(4)或者(5)，就可得到拟设形式的行波解.

2　使用 Tanh 函数法求解带流体动力学阻尼的 IBq 方程

带耗散项的 IBq 方程为

$$u_{tt}-u_{xx}-u_{xxtt}-vu_{xxt}=(u^2)_{xx}. \qquad (6)$$

式中，v 为非零常实数. 把方程(6)行波变换 $\xi=x-ct$ 后，关于 ξ 连续积分两次，积分常数全设为零，得

$$-c^2U''+(c^2-1)U-U^2+cvU'=0\ . \qquad (7)$$

平衡 U''和 U^2 的幂次有

$$m+2=2m,m=2\ .$$

引入独立变量 $y(\xi)=\tanh(k\xi)$，标准 Tanh 函数法有限形式的拟设解为

$$u(x,t)=U(\xi)=S(y)=a_0+a_1y+a_2y^2. \qquad (8)$$

把方程(8)代入方程(7)，整理收集 $y^j(0\leqslant j\leqslant 4)$的系数，并设置为零，得到关于未知系数的非线性方程组：

$$-a_0^2+a_0c^2-a_0-2a_2c^2k^2+a_1vck=0,$$
$$2a_1c^2k^2+a_1c^2+2a_2vck-a_1-2a_0a_1=0,$$
$$-a_1^2-va_1ck+8a_2c^2k^2+a_2c^2-a_2-2a_0a_2=0,$$
$$-2a_1c^2k^2-2a_2vck-2a_1a_2=0,$$
$$-a_2^2-6a_2c^2k^2=0.$$

运用吴方法，结合 Maple 求解这个方程组，得到如下形式的 4 组 8 个解，其中，v 为非零常数.

$$c=\mp\frac{1}{5}\sqrt{-6v^2+25},k=\pm\frac{1}{2}\frac{v}{\sqrt{-6v^2+25}},$$

$$a_0=-\frac{3}{50}v^2,a_1=\frac{3}{25}v^2,a_2=-\frac{3}{50}v^2;$$

$$c=\mp\frac{1}{5}\sqrt{-6v^2+25},k=\pm\frac{1}{2}\frac{v}{\sqrt{-6v^2+25}},$$

$$a_0=-\frac{3}{50}v^2,a_1=-\frac{3}{25}v^2,a_2=-\frac{3}{50}v^2;$$

$$c=\mp\frac{1}{5}\sqrt{6v^2+25},k=\pm\frac{1}{2}\frac{v}{\sqrt{6v^2+25}},$$

$$a_0=\frac{9}{50}v^2,a_1=\frac{3}{25}v^2,a_2=-\frac{3}{50}v^2;$$

$$c=\mp\frac{1}{5}\sqrt{6v^2+25},k=\mp\frac{1}{2}\frac{v}{\sqrt{6v^2+25}},$$

$$a_0=\frac{9}{50}v^2,a_1=-\frac{3}{25}v^2,a_2=-\frac{3}{50}v^2.$$

把这 4 组 8 个解分别代入方程(7),结合独立变量 $y(\xi)=\tanh(k\xi)$,得到原方程的双曲行波解,依次为

$$u_{1,2}(x,t)=-\frac{3}{50}v^2+\frac{3}{50}v^2\tanh k(x-ct)-\frac{3}{50}v^2\tanh^2 k(x-ct),$$

$$u_{3,4}(x,t)=-\frac{3}{50}v^2-\frac{3}{50}v^2\tanh k(x-ct)-\frac{3}{50}v^2\tanh^2 k(x-ct),$$

$$u_{5,6}(x,t)=\frac{9}{50}v^2+\frac{3}{50}v^2\tanh k(x-ct)-\frac{3}{50}v^2\tanh^2 k(x-ct),$$

$$u_{7,8}(x,t)=\frac{9}{50}v^2-\frac{3}{50}v^2\tanh k(x-ct)-\frac{3}{50}v^2\tanh^2 k(x-ct).$$

式中,$u_{1,2}(x,t)$ 分别表示扭结孤立子和反扭结孤立子. 当 c 取正,k 取负时为反扭结孤立子,$v=2$,用 Maple 作解的图像如图 1a 所示;当 c 取负,k 取正时为扭结孤立子,$v=2$,用 Maple 作解的图像如图 1b 所示.

3 使用扩展 Tanh 函数法求解带流体动力学阻尼的 IBq 方程

带耗散项的 IBq 方程为

$$u_{tt}-u_{xx}-u_{xxtt}-vu_{xxt}=(u^2)_{xx}. \tag{9}$$

扩展 Tanh 函数法有限形式的拟设解为

$$u(x,t)=U(\xi)=S(y)=a_2y^2+a_1y+a_0+b_1y^{-1}+b_2y^{-2}. \tag{10}$$

把方程(10)代入方程(7),整理收集 $y^j(-4\leqslant j\leqslant 4)$ 的系数,并设置为零,得到关于未知系数 a_0,a_1,a_2,b_1,b_2,k,c 的非线性方程组:

$$-b_2^2-6b_2c^2k^2=0,$$
$$-2b_1c^2k^2-2b_2vck-2b_1b_2=0,$$
$$-b_1^2-vb_1ck+8b_2c^2k^2+b_2c^2-b_2-2a_0b_2=0,$$
$$2b_1c^2k^2+b_1c^2+2b_2vck-b_1-2a_0b_1-2a_1b_2=0,$$
$$a_0c^2-2a_1b_1-2a_2b_2-a_0-a_0^2-2a_2c^2k^2-2b_2c^2k^2+a_1ckv+b_1ckv=0,$$
$$2a_1c^2k^2+a_1c^2+2a_2vck-a_1-2a_0a_1-2a_2b_1=0,$$
$$-a_1^2-va_1ck+8a_2c^2k^2+a_2c^2-a_2-2a_0a_2=0,$$
$$-2a_2c^2k^2-2a_2vck-2a_1a_2=0,$$
$$-a_2^2-6a_2c^2k^2=0.$$

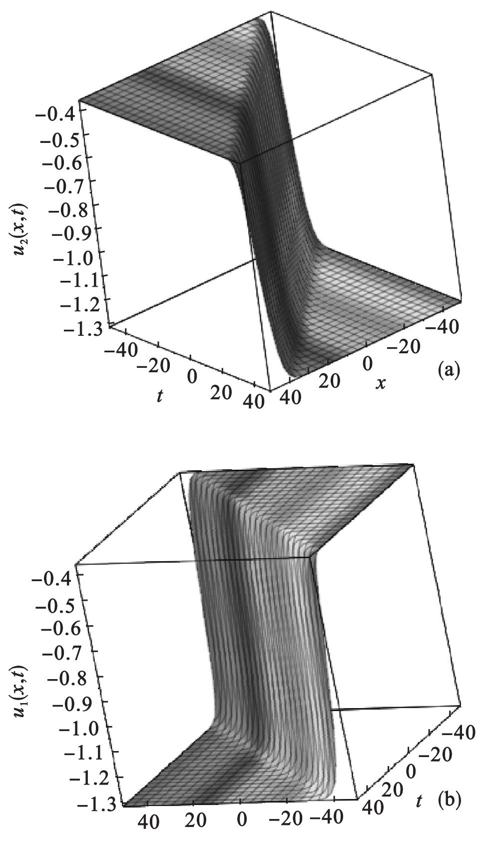

图 1 当 $v=2$ 时,解 $u_{1,2}(x,t)$ 的三维图
Fig. 1 3D plots of $u_{1,2}(x,t)$ when $v=2$
(a)—反扭结孤立子;(b)—扭结孤立子.

运用吴方法,结合 Maple 求解这个方程组,得到如下形式的 4 组 8 个解,其中,v 为非零常数.

$$c=\mp\frac{1}{5}\sqrt{-6v^2+25},k=\mp\frac{1}{4}\frac{v}{\sqrt{-6v^2+25}},$$

$$a_0=-\frac{9}{100}v^2,a_1=-\frac{3}{50}v^2,a_2=-\frac{3}{200}v^2,$$

$$b_1=-\frac{3}{50}v^2,b_2=-\frac{3}{200}v^2;$$

$$c=\mp\frac{1}{5}\sqrt{-6v^2+25},k=\pm\frac{1}{4}\frac{v}{\sqrt{-6v^2+25}},$$

$$a_0 = -\frac{9}{100}v^2, a_1 = \frac{3}{50}v^2, a_2 = -\frac{3}{200}v^2,$$

$$b_1 = \frac{3}{50}v^2, b_2 = -\frac{3}{200}v^2;$$

$$c = \mp\frac{1}{5}\sqrt{6v^2+25}, k = \mp\frac{1}{4}\frac{v}{\sqrt{6v^2+25}},$$

$$a_0 = \frac{3}{20}v^2, a_1 = -\frac{3}{50}v^2, a_2 = -\frac{3}{200}v^2,$$

$$b_1 = -\frac{3}{50}v^2, b_2 = -\frac{3}{200}v^2;$$

$$c = \mp\frac{1}{5}\sqrt{6v^2+25}, k = \pm\frac{1}{4}\frac{v}{\sqrt{6v^2+25}},$$

$$a_0 = \frac{3}{20}v^2, a_1 = \frac{3}{50}v^2, a_2 = -\frac{3}{200}v^2,$$

$$b_1 = \frac{3}{50}v^2, b_2 = -\frac{3}{200}v^2.$$

把这 4 组 8 个解分别代入方程(10),结合独立变量 $y(\xi) = \tanh(k\xi)$,得到原方程的行波解,依次为

$$u_{1,2}(x,t) = -\frac{9}{100}v^2 - \frac{3}{50}v^2\tanh k(x-ct) - \frac{3}{200}v^2\tanh^2 k(x-ct) - \frac{3}{50}v^2\coth k(x-ct) - \frac{3}{200}v^2\coth^2 k(x-ct);$$

$$u_{3,4}(x,t) = -\frac{9}{100}v^2 + \frac{3}{50}v^2\tanh k(x-ct) - \frac{3}{200}v^2\tanh^2 k(x-ct) + \frac{3}{50}v^2\coth k(x-ct) - \frac{3}{200}v^2\coth^2 k(x-ct);$$

$$u_{5,6}(x,t) = \frac{3}{20}v^2 - \frac{3}{50}v^2\tanh k(x-ct) - \frac{3}{200}v^2\tanh^2 k(x-ct) - \frac{3}{50}v^2\coth k(x-ct) - \frac{3}{200}v^2\coth^2 k(x-ct);$$

$$u_{7,8}(x,t) = \frac{3}{20}v^2 + \frac{3}{50}v^2\tanh k(x-ct) - \frac{3}{200}v^2\tanh^2 k(x-ct) + \frac{3}{50}v^2\coth k(x-ct) - \frac{3}{200}v^2\coth^2 k(x-ct).$$

4　结　　论

通过对带流体动力学阻尼的 IBq 方程进行研究发现:虽然对 Bq 方程精确解的研究很多,但 IBq 方程解的研究结果却很少. 在本文中,使用 Tanh 与扩展 Tanh 函数法分别得到了带流体动力学阻尼的 IBq 方程的精确解,在 Maple 的帮助下,过程简单、直接. 带流体动力学阻尼的 IBq 方程行波精确解的获得,为该类方程数值解的进一步研究提供了一定的参考. 这两种方法非常简单,可以应用于许多其他非线性偏微分方程.

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