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三维 Minkowski 空间中伪零曲线的表达形式

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摘　　要: 三维闵可夫斯基(Minkowski)空间中的类空曲线根据其主法向量的性质分为第一类类空曲线、第二类类空曲线和伪零曲线. 讨论了三维 Minkowski 空间中伪零曲线的表达形式. 首先,由伪零曲线的定义给出两个结构函数,并用结构函数将伪零曲线的 Frenet 标架以及曲率函数表达出来,同时找到所定义的两个结构函数之间满足的关系. 最后,讨论曲率函数为常数的伪零曲线及其结构函数的表达形式,并给出相应的例子及图形表示.

关 键 词: Minkowski 空间;伪零曲线;结构函数;曲率函数;表达形式
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Representation Forms of Pseudo Null Curves in the 3-D Minkowski Space

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Abstract: In the 3-D Minkowski space, the spacelike curves can be divided into the spacelike curves of the first kind, the second kind and the pseudo null curves according to the nature of their principal normal vectors. The representation forms of pseudo null curves were surveyed. First, two structure functions were defined by the concept of pseudo null curves. Then the Frenet frames and the curvature functions were expressed by the defined structure functions. At the same time, the relationship between the two structure functions were found. Finally, the representation forms of pseudo null curves with constant curvatures and their structure functions were given, furthermore, the corresponding examples and their graphs were given.

Key words: Minkowski space; pseudo null curve; structure function; curvature function; representation form

设 E_1^3 是三维 Minkowski 空间,其中的内积定义为

$$\langle\ ,\ \rangle = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2.$$

设 E_1^3 中的非零向量 $\boldsymbol{v}$,若 $\langle \boldsymbol{v}, \boldsymbol{v}\rangle > 0$,则称 $\boldsymbol{v}$ 为类空向量;若 $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$,则称 $\boldsymbol{v}$ 为类光向量;若 $\langle \boldsymbol{v}, \boldsymbol{v}\rangle < 0$,则称 $\boldsymbol{v}$ 为类时向量. 特别地,规定零向量为类空向量[1-6].

设 $r(t)$ 是 E_1^3 中任意 1 条正则曲线. 当曲线 $r(t)$ 的切向量为类空向量(类时向量、类光向量)时,称 $r(t)$ 为类空曲线(类时曲线、类光曲线). 特别地,若类空曲线 $r(t)$ 的主法向量为类空向量(类时向量、类光向量),则称其为第一类类空曲线(第二类类空曲线、伪零曲线)[7-9].

2011 年, Liu 等[10]定义了锥曲线的结构函数;2015 年, Qian 等[11]给出了类光曲线的结构函数. 本文用类似的方法描述 E_1^3 中的伪零曲线.

1　预备知识

定义 1[12-13]　设 $r(t)$ 是 E_1^3 中的类空曲线,其 Frenet 标架为 $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$,如果它的主法向量 $\boldsymbol{\beta}$ 与副法向量 $\boldsymbol{\gamma}$ 是线性无关的类光向量,则称 $r(t)$

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为伪零曲线.

引理1[14-15] 设 $r(s):I\to E_1^3$ 是以弧长 s 为参数的伪零曲线,即 $\|\boldsymbol{r}'(s)\|=1$,则其 Frenet 标架 $\{\boldsymbol{r}'(s)=\boldsymbol{\alpha}(s),\boldsymbol{\beta}(s),\boldsymbol{\gamma}(s)\}$ 满足

$$\left.\begin{aligned}&\boldsymbol{\alpha}'(s)=\boldsymbol{\beta}(s),\\&\boldsymbol{\beta}'(s)=\kappa(s)\boldsymbol{\beta}(s),\\&\boldsymbol{\gamma}'(s)=-\boldsymbol{\alpha}(s)-\kappa(s)\boldsymbol{\gamma}(s).\end{aligned}\right\}\tag{1}$$

其中:$\boldsymbol{\alpha}(s)$,$\boldsymbol{\beta}(s)$,$\boldsymbol{\gamma}(s)$ 分别为 $r(s)$ 的切向量、主法向量和副法向量;$\kappa(s)$ 称为曲线 $r(s)$ 的曲率函数.

本文所讨论的曲线为非直线.

2 主要结论

2.1 伪零曲线的结构函数

首先,设伪零曲线 $r(s)$ 的单位切向量为

$$\boldsymbol{r}'(s)=(\xi_1(s),\xi_2(s),\xi_3(s)),$$

因为 $\boldsymbol{r}'(s)$ 是单位类空向量,故满足

$$-\xi_1^2+\xi_2^2+\xi_3^2=1.$$

由 $\xi_3^2-\xi_1^2=1-\xi_2^2$,得

$$\frac{\xi_3+\xi_1}{1+\xi_2}=\frac{1-\xi_2}{\xi_3-\xi_1}=f(s),\xi_2=g(s).$$

这里 $f=f(s)$,$g=g(s)$ 为光滑函数. 显然

$$\left.\begin{aligned}&\xi_1=\frac{1}{2}[f(1+g)-f^{-1}(1-g)],\\&\xi_2=g,\\&\xi_3=\frac{1}{2}[f(1+g)+f^{-1}(1-g)].\end{aligned}\right\}\tag{2}$$

于是,伪零曲线 $r(s)$ 可以表示为

$$r(s)=\frac{1}{2}\int(f(1+g)-f^{-1}(1-g),2g,f(1+g)+f^{-1}(1-g))\mathrm{d}s.$$

因为 $\langle\boldsymbol{r}''(s),\boldsymbol{r}''(s)\rangle=0$,通过计算,可得

$$\frac{f'}{f}=\frac{2g'}{g^2-1}.\tag{3}$$

解微分方程(3)可得

$$g=\frac{C+f}{C-f},\ (C\in\mathbf{R}-\{0\}).$$

因为本文所讨论的曲线为非直线,故 $f'\neq0$,$g'\neq0$.

定理1 设 $r(s)$ 是 E_1^3 中以 s 为弧长参数的伪零曲线,则用函数 $f(s)$,$g(s)$ 可将曲线 $r(s)$ 表示为

$$r(s)=\frac{1}{2}\int(f(1+g)-f^{-1}(1-g),\ 2g,f(1+g)+f^{-1}(1-g))\mathrm{d}s.$$

并且函数 $f=f(s)$,$g=g(s)$ 满足

$$g=\frac{C+f}{C-f},\ (C\in\mathbf{R}-\{0\}).$$

定义2 定理1中的函数 $f(s)$ 和 $g(s)$ 称为伪零曲线 $r(s)$ 的结构函数.

定理2 设 $r(s)$ 是 E_1^3 中以 s 为弧长参数的伪零曲线,其结构函数为 $f(s)$,$g(s)$,则其曲率函数 $\kappa(s)$ 可用结构函数表示为

$$\kappa(s)=\frac{g''(s)}{g'(s)}.\tag{4}$$

它的 Frenet 标架可表示为

$$\left.\begin{aligned}&\boldsymbol{\alpha}=\frac{1}{2}(f(1+g)-f^{-1}(1-g),2g,\\&f(1+g)+f^{-1}(1-g)),\\&\boldsymbol{\beta}=\frac{1}{2}(f'(1+g)+fg'+f'f^{-2}(1-g)+\\&f^{-1}g',2g',f'(1+g)+fg'-\\&f'f^{-2}(1-g)-f^{-1}g'),\\&\boldsymbol{\gamma}=\frac{1}{2g'}(c_1+\int[f^{-1}(1-g)-\\&f(1+g)]\mathrm{d}g,2(c_2-\int g\mathrm{d}g),c_3+\\&\int[f^{-1}(1-g)+f(1+g)]\mathrm{d}g).\end{aligned}\right\}$$

这里,$c_i(i=1,2,3)\in\mathbf{R}$.

证明 由引理1及式(2)易得 $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$ 的表达形式. 且有

$$\boldsymbol{r}'''(s)=\frac{1}{2}((f'(1+g)+fg'+f'f^{-2}(1-g)+f^{-1}g')',2g'',(f'(1+g)+fg'-f'f^{-2}(1-g)-f^{-1}g')').$$

由 $\boldsymbol{r}'''=\boldsymbol{\alpha}''=\kappa\boldsymbol{r}''$,可得 $\kappa(s)=\dfrac{g''(s)}{g'(s)}$,进一步由 $\boldsymbol{\gamma}'=-\boldsymbol{\alpha}-\kappa\boldsymbol{\gamma}$,通过解微分方程可得向量 $\boldsymbol{\gamma}$ 的表达形式.

2.2 常曲率伪零曲线

定理3 设 $r(s)$ 是 E_1^3 中以 s 为弧长参数的伪零曲线,如果 $r(s)$ 的曲率函数 $\kappa(s)$ 为常数,则它的结构函数 $f(s)$,$g(s)$ 为

1)当 $\kappa=c=0$ 时,其结构函数

$$f(s)=\frac{as-1}{as+1},\ g(s)=as,$$

这里 $a\in\mathbf{R}-\{0\}$;

2)当 $\kappa=c\neq0$ 时,其结构函数

$$f(s)=\frac{\mathrm{e}^{cs}-1}{\mathrm{e}^{cs}+1},\ g(s)=\mathrm{e}^{cs}.$$

证明 设曲率函数 $\kappa(s)$ 为常数,即 $\kappa=c$,($c\in\mathbf{R}$). 由式(4)得

$$\frac{g''(s)}{g'(s)}=c.$$

1）当 $\kappa(s)=c=0$ 时，有

$g'(s)=a, g(s)=as+c_1, (a\in\mathbf{R}-\{0\}, c_1\in\mathbf{R})$. 不失一般性，通过参数变换可以省略积分常数 c_1，则 $g(s)=as$. 由式(3)可得

$$\frac{f'(s)}{f(s)}=\frac{2a}{a^2s^2-1}. \tag{5}$$

解微分方程(5)得

$$c_2 f(s)=\frac{as-1}{as+1}, (c_2\in\mathbf{R}-\{0\}).$$

通过适当变换，可令 $c_2=1$，于是得

$$f(s)=\frac{as-1}{as+1}.$$

2）当 $\kappa(s)=c\neq 0$ 时，与 1）的推导类似，有

$$g(s)=\mathrm{e}^{cs}, f(s)=\frac{\mathrm{e}^{cs}-1}{\mathrm{e}^{cs}+1}.$$

由定理 1 和定理 3，显然有下面的结论.

定理 4　设 $r(s)$ 是 E_1^3 中的常曲率伪零曲线，则 $r(s)$ 可表示为

1）当 $\kappa=c=0$ 时，

$$r(s)=\frac{1}{2}(as^2, as^2, -2s),$$

这里 $a\in\mathbf{R}-\{0\}$；

2）当 $\kappa=c\neq 0$ 时，

$$r(s)=\frac{1}{c}(\mathrm{e}^{cs}, \mathrm{e}^{cs}, -cs).$$

例 1　设定理 3 中曲率 $\kappa(s)=0$ 的伪零曲线的结构函数 $f(s)=\frac{2s-1}{2s+1}$, $g(s)=2s$, 则曲线 $r_1(s)=(s^2, s^2, -s)$，如图 1 所示.

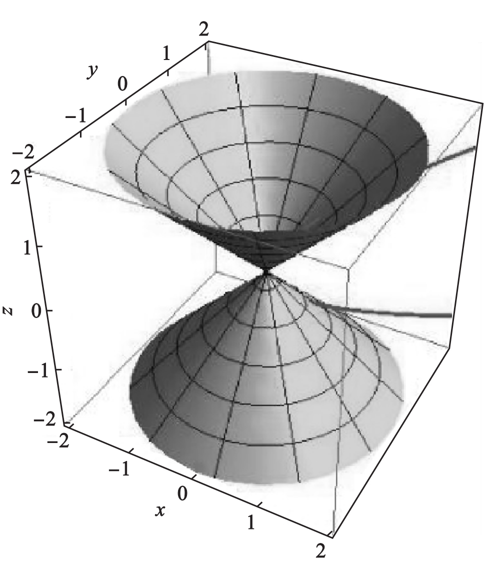

图 1　伪零曲线 $r_1(s)$ 的光锥面表示

Fig. 1　Pseudo null curve $r_1(s)$ shown with the lightlike cone

例 2　设定理 3 中曲率 $\kappa(s)=c\neq 0$ 的伪零曲线的结构函数 $f(s)=\frac{\mathrm{e}^s-1}{\mathrm{e}^s+1}$, $g(s)=\mathrm{e}^s$, 则曲线 $r_2(s)=(e^s, e^s, -s)$，如图 2 所示.

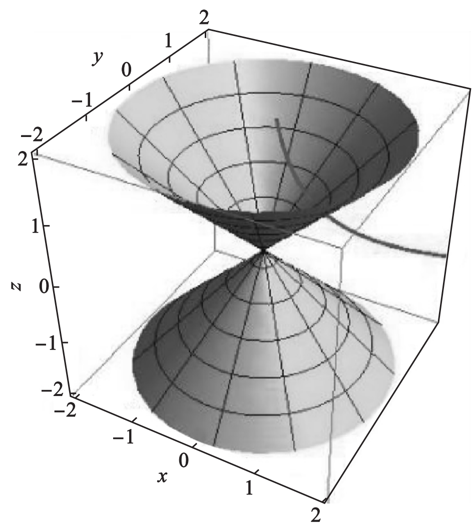

图 2　伪零曲线 $r_2(s)$ 的光锥面表示

Fig. 2　Pseudo null curve $r_2(s)$ shown with the lightlike cone

3　结　　语

本文定义了伪零曲线的结构函数，并用结构函数表示了伪零曲线及其曲率函数. 讨论了曲率函数为常数的伪零曲线的结构函数和曲线表达式，并且给出了相应的例子. 为今后伪零曲线的进一步研究提供了新的思路和方法.

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