

区间时变时滞线性系统的指数稳定性

郑连伟, 宋叔尼

(东北大学 理学院, 辽宁 沈阳 110819)

摘 要: 基于 Lyapunov 泛函方法, 研究时滞是时变的且属于一个区间的线性时滞系统的指数稳定性. 以线性矩阵不等式的形式给出了一个新的指数稳定性判别准则. 在估计 Lyapunov 泛函的导数时, 利用凸组合和倒数凸组合方法得到了较小的上界, 从而使得到的指数稳定性条件具有较小的保守性. 另外, 所得状态变量的指数上界只依赖于初始函数本身而不涉及其导数. 最后, 用数值算例验证了所得方法的有效性.

关 键 词: 时变时滞; 指数稳定性; 线性矩阵不等式; 倒数凸组合; Lyapunov 泛函

中图分类号: TP 13 **文献标志码:** A **文章编号:** 1005-3026(2014)09-1225-04

Exponential Stability for Linear Systems with Interval Time-Varying Delays

ZHENG Lian-wei, SONG Shu-ni

(School of Sciences, Northeastern University, Shenyang 110819, China. Corresponding author: ZHENG Lian-wei, E-mail: zhenglianwei@sina.com)

Abstract: Based on the Lyapunov functional, exponential stability for linear systems with time-varying delay was studied. The time delay is a differentiable function belonging to a given interval. A new criterion for exponential stability was proposed in the form of linear matrix inequalities. When the derivative of the Lyapunov functional was estimated, a tighter upper bound was obtained using convex combination and reciprocally convex combination approaches that led to less conservatism of the condition for exponential stability. In addition, the obtained exponential upper bound of the state depended only on the initial function itself. An example was then presented to show the effectiveness of the proposed method.

Key words: time-varying delay; exponential stability; linear matrix inequalities; reciprocally convex combination; Lyapunov functional

时滞系统的稳定性分析是长期受到关注的一个研究领域, 近年来已取得很多成果. 指数稳定性分析是要确定系统解的指数上界, 从而可以估计解的收敛速率. 文献[1-3]和文献[4-5]分别给出了依赖于时滞上界和依赖于时滞区间的指数稳定性分析方法. 文献[4-5]得到的指数上界涉及初始函数的导数, 需要额外要求初始函数是可导的. 文献[3-4]由于对 Lyapunov 泛函的导数估计得不够精确而增大了结果的保守性. 本文针对时滞属于一个区间的时滞系统给出一种新的指数稳定性分析方法. 该方法同时采用凸组合及倒数凸组合方法估计 Lyapunov 泛函的导数, 给出的导

数上界小于以往文献, 因而所得指数稳定性条件具有较小的保守性, 而且状态变量的指数上界只依赖于初始函数本身而不涉及其导数.

1 问题描述与预备知识

本文用 $\lambda_{\max}(\cdot)$ 和 $\lambda_{\min}(\cdot)$ 分别表示对称矩阵的最大和最小特征值; 用 $\|\cdot\|$ 表示向量的欧几里德范数或其诱导的矩阵范数; 对于区间 $[-h_2, 0]$ 上的连续向量值函数 $\varphi(t)$, 其范数定义为 $\|\varphi\|_c = \max_{-h_2 \leq t \leq 0} \|\varphi(t)\|$.

考虑如下时滞系统:

$$\left. \begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{x}(t-d(t)), \\ \boldsymbol{x}(t) &= \boldsymbol{\varphi}(t), t \in [-h_2, 0]. \end{aligned} \right\} \quad (1)$$

式中: $\boldsymbol{x}(t) \in \mathbf{R}^n$ 为状态向量; $\boldsymbol{A}, \boldsymbol{B} \in \mathbf{R}^{n \times n}$ 是常数矩阵; $\boldsymbol{\varphi}(t)$ 是初始函数; $d(t)$ 是可微分的时变时滞函数, 满足

$$0 \leq h_1 \leq d(t) \leq h_2, \dot{d}(t) \leq \mu < 1. \quad (2)$$

式中, h_1, h_2, μ 是常数.

系统的指数稳定性由以下定义给出.

定义 1 如果存在数量 $\sigma > 0$ 和 $\alpha > 0$, 使得 $\|\boldsymbol{x}(t)\| \leq \alpha \|\boldsymbol{\varphi}\| e^{-\sigma t}$, 则称系统按衰减率 α 指数稳定.

下面的引理 1 给出了 Jensen 积分不等式, 引理 2 是关于倒数凸组合最小值的一个结果.

引理 1^[6] 设 $\boldsymbol{M} > 0$ 是正定矩阵, $\boldsymbol{\omega}(s)$ 是 $(-\infty, +\infty)$ 上的向量值连续函数, 则有

1) 设 $a \leq b$ 是两个数量, 则

$$\left(\int_a^b \boldsymbol{\omega}^T(s) ds \right) \boldsymbol{M} \left(\int_a^b \boldsymbol{\omega}(s) ds \right) \leq (b-a) \int_a^b \boldsymbol{\omega}^T(s) \times \boldsymbol{M} \boldsymbol{\omega}(s) ds;$$

2) 设 a, b, c 是数量, 且 $a \geq b \geq 0$, 则

$$\left(\int_{-a}^{-b} \int_{c+\theta}^c \boldsymbol{\omega}^T(s) ds \right) \boldsymbol{M} \left(\int_{-a}^{-b} \int_{c+\theta}^c \boldsymbol{\omega}(s) ds \right) \leq \frac{a^2 - b^2}{2} \int_{-a}^{-b} \int_{c+\theta}^c \boldsymbol{\omega}^T(s) \boldsymbol{M} \boldsymbol{\omega}(s) ds.$$

引理 2^[7] 设 $\begin{bmatrix} \boldsymbol{M} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{N} \end{bmatrix} \geq 0$ 是半正定矩阵, 则

对任何 $0 < \alpha < 1$, 有

$$\begin{bmatrix} \frac{1}{\alpha} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{1-\alpha} \boldsymbol{N} \end{bmatrix} \geq \begin{bmatrix} \boldsymbol{M} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{N} \end{bmatrix}.$$

2 主要结果

为判断指数稳定性, 首先引入以下变量:

$$\boldsymbol{z}(t) = e^{\alpha t} \boldsymbol{x}(t). \quad (3)$$

由式(1)可知 $\boldsymbol{z}(t)$ 满足以下方程:

$$\left. \begin{aligned} \dot{\boldsymbol{z}}(t) &= (\alpha \boldsymbol{I} + \boldsymbol{A}) \boldsymbol{z}(t) + e^{\alpha d(t)} \boldsymbol{B} \boldsymbol{z}(t-d(t)), \\ \boldsymbol{z}(t) &= \boldsymbol{\psi}(t), t \in [-h_2, 0]. \end{aligned} \right\} \quad (4)$$

式中,

$$\boldsymbol{\psi}(t) = e^{\alpha t} \boldsymbol{\varphi}(t). \quad (5)$$

引入以下增广向量:

$$\boldsymbol{\xi}(t) = \text{col} \{ \boldsymbol{z}(t), \boldsymbol{z}(t-d(t)), \boldsymbol{z}(t-h_1), \boldsymbol{z}(t-h_2) \},$$

$$\dot{\boldsymbol{z}}(t-h_1), \dot{\boldsymbol{z}}(t-h_2), \int_{t-h_1}^t \boldsymbol{z}(s) ds,$$

$$\int_{t-d(t)}^{t-h_1} \boldsymbol{z}(s) ds, \int_{t-h_2}^{t-d(t)} \boldsymbol{z}(s) ds, \int_{-h_1}^0 \int_{t+\theta}^t \boldsymbol{z}(s) ds d\theta,$$

$$\int_{-h_2}^{-h_1} \int_{t+\theta}^t \boldsymbol{z}(s) ds d\theta, \dot{\boldsymbol{z}}(t) \}. \quad (6)$$

引入与 $\boldsymbol{\xi}(t)$ 相对应的分块矩阵:

$$\boldsymbol{e}_i^T = [0 \quad \cdots \quad 0 \quad \boldsymbol{I} \quad 0 \quad \cdots \quad 0], i = 1, \dots, 12.$$

下面的定理给出了一个新的指数稳定性准则.

定理 1 考虑系统(1)和约束(2), 给定正数 $h_1 \leq h_2, \alpha$, 如果存在正定矩阵 $\boldsymbol{P}, \boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{Z}_1, \boldsymbol{Z}_2$ 和适当维数的矩阵 $\boldsymbol{S}_1, \boldsymbol{S}_2$ 使得以下三个线性矩阵不等式成立:

$$\begin{aligned} \boldsymbol{\Omega}(h_k) &= \boldsymbol{\Omega}_0 + \boldsymbol{S}_2 (\alpha \boldsymbol{I} + \boldsymbol{A}) \boldsymbol{e}_1^T + \boldsymbol{e}_1 (\alpha \boldsymbol{I} + \boldsymbol{A})^T \boldsymbol{S}_2^T + \\ &e^{\alpha h_k} \boldsymbol{S}_2 \boldsymbol{B} \boldsymbol{e}_2^T + e^{\alpha h_k} \boldsymbol{e}_2 \boldsymbol{B}^T \boldsymbol{S}_2^T - \boldsymbol{S}_2 \boldsymbol{e}_{12}^T - \boldsymbol{e}_{12} \boldsymbol{S}_2^T < 0, k = 1, 2. \end{aligned} \quad (7)$$

$$\begin{bmatrix} \boldsymbol{Z}_2 & \boldsymbol{S}_1 \\ \boldsymbol{S}_1^T & \boldsymbol{Z}_2 \end{bmatrix} \geq 0, \quad (8)$$

则系统按衰减率 α 指数稳定.

式中,

$$\begin{aligned} \boldsymbol{\Omega}_0 &= \boldsymbol{C} \boldsymbol{P} \boldsymbol{D}^T + \boldsymbol{D} \boldsymbol{P} \boldsymbol{C}^T + \boldsymbol{K} (\boldsymbol{Q}_2 - \boldsymbol{Q}_1) \boldsymbol{K}^T - \boldsymbol{L} \boldsymbol{Q}_2 \boldsymbol{L}^T + \\ &\boldsymbol{e}_1 \boldsymbol{Q}_3 \boldsymbol{e}_1^T - (1 - \mu) \boldsymbol{e}_2 \boldsymbol{Q}_3 \boldsymbol{e}_2^T - \boldsymbol{W} \boldsymbol{Z}_1 \boldsymbol{W}^T - \\ &[\boldsymbol{M} \quad \boldsymbol{N}] \begin{bmatrix} \boldsymbol{Z}_2 & \boldsymbol{S}_1 \\ \boldsymbol{S}_1^T & \boldsymbol{Z}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{M}^T \\ \boldsymbol{N}^T \end{bmatrix} - \boldsymbol{T} \boldsymbol{R}_1 \boldsymbol{T}^T - \boldsymbol{H} \boldsymbol{R}_2 \boldsymbol{H}^T + \\ &\boldsymbol{F} (\boldsymbol{Q}_1 + h_1^2 \boldsymbol{Z}_1 + h_{12}^2 \boldsymbol{Z}_2 + \frac{h_1^4}{4} \boldsymbol{R}_1 + h_s^2 \boldsymbol{R}_2) \boldsymbol{F}^T. \end{aligned} \quad (9)$$

式中: $\boldsymbol{C} = [\boldsymbol{e}_1 \quad \boldsymbol{e}_3 \quad \boldsymbol{e}_4 \quad \boldsymbol{e}_7 \quad \boldsymbol{e}_8 + \boldsymbol{e}_9 \quad \boldsymbol{e}_{10} \quad \boldsymbol{e}_{11}]$; $\boldsymbol{D} = [\boldsymbol{e}_{12} \quad \boldsymbol{e}_5 \quad \boldsymbol{e}_6 \quad \boldsymbol{e}_1 - \boldsymbol{e}_3 \quad \boldsymbol{e}_3 - \boldsymbol{e}_4 \quad h_1 \boldsymbol{e}_1 - \boldsymbol{e}_7 \quad h_{12} \boldsymbol{e}_1 - \boldsymbol{e}_8 - \boldsymbol{e}_9]$;

$\boldsymbol{K} = [\boldsymbol{e}_3 \quad \boldsymbol{e}_5]$; $\boldsymbol{L} = [\boldsymbol{e}_4 \quad \boldsymbol{e}_6]$; $\boldsymbol{W} = [\boldsymbol{e}_7 \quad \boldsymbol{e}_1 - \boldsymbol{e}_3]$;

$\boldsymbol{M} = [\boldsymbol{e}_8 \quad \boldsymbol{e}_3 - \boldsymbol{e}_2]$; $\boldsymbol{N} = [\boldsymbol{e}_9 \quad \boldsymbol{e}_2 - \boldsymbol{e}_4]$;

$\boldsymbol{T} = [\boldsymbol{e}_{10} \quad h_1 \boldsymbol{e}_1 - \boldsymbol{e}_7]$; $\boldsymbol{H} = [\boldsymbol{e}_{11} \quad h_{12} \boldsymbol{e}_1 - \boldsymbol{e}_8 - \boldsymbol{e}_9]$;

$\boldsymbol{F} = [\boldsymbol{e}_1 \quad \boldsymbol{e}_{12}]$; $h_{12} = h_2 - h_1$; $h_s = \frac{h_2^2 - h_1^2}{2}$.

证明 定义如下两个增广向量:

$$\boldsymbol{\zeta}(t) = \text{col} \{ \boldsymbol{z}(t), \boldsymbol{z}(t-h_1), \boldsymbol{z}(t-h_2), \int_{t-h_1}^t \boldsymbol{z}(s) ds,$$

$$\int_{t-h_2}^{t-h_1} \boldsymbol{z}(s) ds, \int_{-h_1}^0 \int_{t+\theta}^t \boldsymbol{z}(s) ds d\theta, \int_{-h_2}^{-h_1} \int_{t+\theta}^t \boldsymbol{z}(s) ds d\theta \}; \quad (10)$$

$$\boldsymbol{y}^T(t) = [\boldsymbol{z}^T(t) \quad \dot{\boldsymbol{z}}^T(t)]. \quad (11)$$

选取 Lyapunov 泛函:

$$\boldsymbol{V}(t) = \sum_{k=1}^5 \boldsymbol{V}_k(t), t \geq h_2.$$

式中:

$$\boldsymbol{V}_1(t) = \boldsymbol{\zeta}^T(t) \boldsymbol{P} \boldsymbol{\zeta}(t) + \int_{t-h_1}^t \boldsymbol{y}^T(s) \boldsymbol{Q}_1 \boldsymbol{y}(s) ds +$$

$$\int_{t-h_2}^{t-h_1} \boldsymbol{y}^T(s) \boldsymbol{Q}_2 \boldsymbol{y}(s) ds + \int_{t-d(t)}^t \boldsymbol{z}^T(s) \boldsymbol{Q}_3 \boldsymbol{z}(s) ds;$$

$$\boldsymbol{V}_2(t) = h_1 \int_{-h_1}^0 \int_{t+\theta}^t \boldsymbol{y}^T(s) \boldsymbol{Z}_1 \boldsymbol{y}(s) ds d\theta;$$

$$V_3(t) = h_{12} \int_{-h_2}^{-h_1} \int_{t+\theta}^t \mathbf{y}^T(s) \mathbf{Z}_2 \mathbf{y}(s) ds d\theta;$$

$$V_4(t) = \frac{h_1^2}{2} \int_{-h_1}^0 \int_{t+\lambda}^0 \mathbf{y}^T(s) \mathbf{R}_1 \mathbf{y}(s) ds d\lambda d\theta;$$

$$V_5(t) = h_s \int_{-h_2}^{-h_1} \int_{t+\lambda}^0 \mathbf{y}^T(s) \mathbf{R}_2 \mathbf{y}(s) ds d\lambda d\theta.$$

求 $V_k(t)$ 沿系统(1)的轨线的导数,得到

$$\begin{aligned} \dot{V}_1(t) = & \xi^T(t) (\mathbf{CPD}^T + \mathbf{DPC}^T) \xi(t) + \mathbf{y}^T(t) \mathbf{Q}_1 \times \\ & \mathbf{y}(t) - \mathbf{y}^T(t-h_1) \mathbf{Q}_1 \mathbf{y}(t-h_1) + \mathbf{y}^T(t-h_1) \times \\ & \mathbf{Q}_2 \mathbf{y}(t-h_1) - \mathbf{y}^T(t-h_2) \mathbf{Q}_2 \mathbf{y}(t-h_2) + \\ & \mathbf{z}^T(t) \mathbf{Q}_3 \mathbf{z}(t) - (1-d(t)) \mathbf{z}^T(t-d(t)) \times \\ & \mathbf{Q}_3 \mathbf{z}(t-d(t)); \end{aligned}$$

$$\dot{V}_2(t) = h_1^2 \mathbf{y}^T(t) \mathbf{Z}_1 \mathbf{y}(t) - h_1 \int_{t-h_1}^t \mathbf{y}^T(s) \mathbf{Z}_1 \mathbf{y}(s) ds;$$

$$\dot{V}_3(t) = h_{12}^2 \mathbf{y}^T(t) \mathbf{Z}_2 \mathbf{y}(t) - h_{12} \int_{t-h_2}^{t-h_1} \mathbf{y}^T(s) \mathbf{Z}_2 \mathbf{y}(s) ds;$$

$$\dot{V}_4(t) = \frac{h_1^4}{4} \mathbf{y}^T(t) \mathbf{R}_1 \mathbf{y}(t) - \frac{h_1^2}{2} \int_{-h_1}^0 \int_{t+\theta}^0 \mathbf{y}^T(s) \mathbf{R}_1 \mathbf{y}(s) ds d\theta;$$

$$\dot{V}_5(t) = h_s^2 \mathbf{y}^T(t) \mathbf{R}_2 \mathbf{y}(t) - h_s \int_{-h_2}^{-h_1} \int_{t+\theta}^0 \mathbf{y}^T(s) \mathbf{R}_2 \mathbf{y}(s) ds d\theta.$$

对 $\dot{V}_2(t), \dot{V}_4(t), \dot{V}_5(t)$ 中的积分应用引理1,可得

$$h_1 \int_{t-h_1}^t \mathbf{y}^T(s) \mathbf{Z}_1 \mathbf{y}(s) ds \geq \xi^T(t) \mathbf{WZ}_1 \mathbf{W}^T \xi(t);$$

$$\frac{h_1^2}{2} \int_{-h_1}^0 \int_{t+\theta}^0 \mathbf{y}^T(s) \mathbf{R}_1 \mathbf{y}(s) ds d\theta \geq \xi^T(t) \mathbf{TR}_1 \mathbf{T}^T \xi(t);$$

$$h_s \int_{-h_2}^{-h_1} \int_{t+\theta}^0 \mathbf{y}^T(s) \mathbf{R}_2 \mathbf{y}(s) ds d\theta \geq \xi^T(t) \mathbf{HR}_2 \mathbf{H}^T \xi(t).$$

把 $\dot{V}_3(t)$ 中的积分写成两个积分之和,应用引理1,再由式(8)和引理2得到

$$\begin{aligned} h_{12} \int_{t-h_2}^{t-h_1} \mathbf{y}^T(s) \mathbf{Z}_2 \mathbf{y}(s) ds &= h_{12} \int_{t-d(t)}^{t-h_1} \mathbf{y}^T(s) \mathbf{Z}_2 \mathbf{y}(s) ds + \\ & h_{12} \int_{t-h_2}^{t-d(t)} \mathbf{y}^T(s) \mathbf{Z}_2 \mathbf{y}(s) ds \geq \\ & \xi^T(t) [\mathbf{M} \quad \mathbf{N}] \begin{bmatrix} \mathbf{Z}_2 & \mathbf{S}_1 \\ \mathbf{S}_1^T & \mathbf{Z}_2 \end{bmatrix} \times \\ & \begin{bmatrix} \mathbf{M}^T \\ \mathbf{N}^T \end{bmatrix} \xi(t). \end{aligned}$$

利用这几个不等式可得

$$\dot{V}(t) \leq \xi^T(t) \mathbf{\Omega}_0 \xi(t). \quad (12)$$

利用式(6),式(4)中第一个等式可以写成

$$(\alpha \mathbf{I} + \mathbf{A}) \mathbf{e}_1^T \xi(t) + \mathbf{e}^{\alpha d(t)} \mathbf{B} \mathbf{e}_2^T \xi(t) - \mathbf{e}_{12}^T \xi(t) = 0.$$

对任何适当维数的矩阵 \mathbf{S}_2 ,有

$$2\xi^T \mathbf{S}_2 [(\alpha \mathbf{I} + \mathbf{A}) \mathbf{e}_1^T \xi + \mathbf{e}^{\alpha d(t)} \mathbf{B} \mathbf{e}_2^T \xi - \mathbf{e}_{12}^T \xi] = 0.$$

把这个等式的左边加到式(12)的右边,得到

$$\dot{V}(t) \leq \xi^T(t) \mathbf{\Omega}(d(t)) \xi(t).$$

式中,

$$\begin{aligned} \mathbf{\Omega}(d(t)) = & \mathbf{\Omega}_0 + \mathbf{S}_2 (\alpha \mathbf{I} + \mathbf{A}) \mathbf{e}_1^T + \mathbf{e}_1 (\alpha \mathbf{I} + \mathbf{A})^T \mathbf{S}_2^T + \\ & \mathbf{e}^{\alpha d(t)} \mathbf{S}_2 \mathbf{B} \mathbf{e}_2^T + \mathbf{e}^{\alpha d(t)} \mathbf{e}_2 \mathbf{B}^T \mathbf{S}_2^T - \mathbf{S}_2 \mathbf{e}_{12}^T - \mathbf{e}_{12} \mathbf{S}_2^T. \end{aligned}$$

因为 $\mathbf{\Omega}(d(t))$ 关于 $\mathbf{e}^{\alpha d(t)}$ 是仿射的,所以由式(7)可得对任何 $h_1 \leq d(t) \leq h_2$,有 $\mathbf{\Omega}(d(t)) < 0$,因此

$$\dot{V}(t) \leq -\delta \|\mathbf{z}(t)\|^2. \quad (13)$$

式中, $\delta = \min \{ \lambda_{\min}(-\mathbf{\Omega}(d)), d \in [h_1, h_2] \} > 0$.

由式(4),可得

$$\begin{aligned} 2\mathbf{z}^T(t) \dot{\mathbf{z}}(t) = & 2\mathbf{z}^T(t) (\alpha \mathbf{I} + \mathbf{A}) \mathbf{z}(t) + \\ & 2\mathbf{e}^{\alpha d(t)} \mathbf{z}^T(t) \mathbf{B} \mathbf{z}(t-d(t)). \end{aligned}$$

利用式(2),可得

$$\frac{d}{dt} \|\mathbf{z}(t)\|^2 \leq k_1 \|\mathbf{z}(t)\|^2 + k_2 \|\mathbf{z}(t-d(t))\|^2.$$

式中, $k_1 = 2 \|\alpha \mathbf{I} + \mathbf{A}\| + \mathbf{e}^{\alpha h_2}$, $k_2 = \|\mathbf{B}^T \mathbf{B}\| \mathbf{e}^{\alpha h_2}$.

把该不等式从0到t积分,并做变量代换 $v = s - d(s)$,可得

$$\|\mathbf{z}(t)\|^2 \leq \|\mathbf{z}(0)\|^2 + k_3 \int_{-h_2}^t \|\mathbf{z}(v)\|^2 dv.$$

式中, $k_3 = k_1 + \frac{k_2}{1-\mu}$.

对于 $t \geq 0$,由这个不等式可得

$$\|\mathbf{z}(t)\|^2 \leq (1 + k_3 h_2) \|\boldsymbol{\psi}\|_c^2 + k_3 \int_0^t \|\mathbf{z}(v)\|^2 dv,$$

再由 Gronwall - Bellman 不等式^[8],可得

$$\|\mathbf{z}(t)\|^2 \leq (1 + k_3 h_2) \|\boldsymbol{\psi}\|_c^2 \mathbf{e}^{k_3 t}. \quad (14)$$

由不等式(14)和不等式

$$\left\| \int_a^b \mathbf{z}(s) ds \right\| \leq \int_a^b \|\mathbf{z}(s)\| ds$$

可得

$$\|\boldsymbol{\zeta}(h_2)\|^2 \leq k_4 \|\boldsymbol{\Psi}\|_c^2. \quad (15)$$

式中, $k_4 = (1 + k_3 h_2) [\mathbf{e}^{k_3 h_2} + \mathbf{e}^{k_3(h_2-h_1)} + 1 + (2 + h_1^2 + h_{12}^2) h_2^2 \mathbf{e}^{k_3 h_2}]$.

由式(11)和式(4),可得

$$\|\mathbf{y}(s)\|^2 \leq k_5 \|\mathbf{z}(s)\|^2 + k_6 \|\mathbf{z}(s-d(s))\|^2.$$

式中, $k_5 = 1 + 2 \|\alpha \mathbf{I} + \mathbf{A}\|^2$, $k_6 = 2\mathbf{e}^{2\alpha h_2} \|\mathbf{B}\|^2$.

把该不等式从0到t积分,并做变量代换 $v = s - d(s)$,可得

$$\begin{aligned} \int_0^{h_2} \|\mathbf{y}(s)\|^2 ds &\leq \left(k_5 + \frac{k_6}{1-\mu} \right) \int_0^{h_2} \|\mathbf{z}(s)\|^2 ds + \\ & \frac{k_6}{1-\mu} \int_{-h_2}^0 \|\mathbf{z}(v)\|^2 dv. \end{aligned}$$

利用式(14),可进一步得到

$$\int_0^{h_2} \|\mathbf{y}(s)\|^2 ds \leq k_7 \|\boldsymbol{\psi}\|_c^2. \quad (16)$$

式中, $k_7 = \left(k_5 + \frac{k_6}{1-\mu} \right) (1 + k_3 h_2) h_2 \mathbf{e}^{k_3 h_2} + \frac{k_6 h_2}{1-\mu}$.

利用式(15)和式(16),可得

$$V(h_2) = \sum_{k=1}^5 V_k(h_2) \leq k_8 \|\boldsymbol{\psi}\|_c^2. \quad (17)$$

式中, $k_8 = \lambda_{\max}(\boldsymbol{P})k_4 + (\lambda_{\max}(\boldsymbol{Q}_1) + \lambda_{\max}(\boldsymbol{Q}_2) + \lambda_{\max}(\boldsymbol{Q}_3) + \lambda_{\max}(\boldsymbol{Z}_1)h_1^2 + \lambda_{\max}(\boldsymbol{Z}_1)h_{12}^2 + \lambda_{\max}(\boldsymbol{R}_1) \times \frac{h_1^4}{4} + \lambda_{\max}(\boldsymbol{R}_2)h_s^2)k_7$.

由式(13)及 $V(t)$ 的定义有
 $\lambda_{\min}(\boldsymbol{P})\|\boldsymbol{z}(t)\|^2 \leq V(t) \leq V(h_2), t \geq h_2$. (18)
从而由式(17)和式(18)可得

$$\|\boldsymbol{z}(t)\| \leq \sqrt{\frac{k_8}{\lambda_{\min}(\boldsymbol{P})}} \|\boldsymbol{\psi}\|_c, t \geq h_2.$$

由式(3)和式(5)可得

$$\|\boldsymbol{x}(t)\| \leq \sqrt{\frac{k_8}{\lambda_{\min}(\boldsymbol{P})}} \|\boldsymbol{\varphi}\|_c e^{-\alpha t}, t \geq h_2.$$

由式(14), 式(3)和式(5)可得

$$\|\boldsymbol{x}(t)\| \leq \sqrt{1 + k_3 h_2} e^{0.5 k_3 h_2} \|\boldsymbol{\varphi}\|_c e^{-\alpha t}, 0 \leq t \leq h_2.$$

表 1 对应于给定衰减率 α 的时滞上界和下界
Table 1 Upper and lower delay bound for given decay rate α

方法	α				
	0.1	0.3	0.5	0.7	0.9
文献[2]	0 ~ 11.455	0 ~ 3.442	0 ~ 1.920	0 ~ 1.318	0 ~ 0.952
文献[3]	0 ~ 1.444	0 ~ 1.055	0 ~ 0.831	0 ~ 0.674	0 ~ 0.527
文献[4]	11.452 ~ 11.455	3.440 ~ 3.442	1.918 ~ 1.920	1.298 ~ 1.317	0.821 ~ 0.952
文献[5]	11.563 ~ 11.563	3.718 ~ 3.736	2.247 ~ 2.269	1.663 ~ 1.665	1.186 ~ 1.191
定理 1	5.534 ~ 11.573	1.973 ~ 3.768	0.999 ~ 2.292	0.919 ~ 1.683	0 ~ 1.191

4 结 语

本文研究了区间时变时滞线性系统的指数稳定性问题. 利用 Lyapunov 泛函方法, 以线性矩阵不等式的形式, 给出了系统指数稳定的判别条件. 由于同时采用凸组合及倒数凸组合方法估计 Lyapunov 泛函的导数, 使得该指数稳定性条件具有较小的保守性. 改进 Lyapunov 泛函的结构, 寻找更好的导数估计方法是进一步减小保守性需要研究的问题.

参考文献:

[1] Xu S, Lam J, Zhong M Y. New exponential estimates for time-delay systems [J]. *IEEE Transactions on Automatic Control*, 2006, 51(9): 1501 – 1505.
[2] Kwon O M, Park J H, Lee S M. Exponential stability for uncertain dynamic systems with time-varying delays: an LMI optimization approach [J]. *Journal of Optimization Theory*

综合以上两个不等式可知, 对任意 $t > 0$, 有
 $\|\boldsymbol{x}(t)\| \leq \max\left(\sqrt{\frac{k_8}{\lambda_{\min}(\boldsymbol{P})}}, \sqrt{1 + k_3 h_2} e^{0.5 k_3 h_2}\right) \times \|\boldsymbol{\varphi}\|_c e^{-\alpha t}$,
这就证明了系统按衰减率 α 指数稳定.

3 数值算例

考虑系统(1), 其中
 $A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -0.5 & 0.1 \\ 0.3 & 0 \end{bmatrix}$.
取 $\mu = 0$, 对不同的衰减率 α , 利用定理 1 和一些文献的方法求出时滞的上界和下界, 列于表 1 进行比较, 可见本文方法得到了较大的时滞上界和较长的时滞区间, 因而具有优越性.

and Applications, 2008, 137(3): 521 – 532.
[3] Hien L V, Phat V N. Exponential stability and stabilization of a class of uncertain linear time-delay systems [J]. *Journal of Franklin Institute*, 2009, 346(6): 611 – 625.
[4] Phat V N, Khongtham Y, Ratchagit K. LMI approach to exponential stability of linear systems with interval time-varying delays [J]. *Linear Algebra and Its Applications*, 2012, 436(1): 243 – 251.
[5] Guo H D, Gu H, Xing J, et al. Asymptotic and exponential stability of uncertain system with interval delay [J]. *Applied Mathematics and Computation*, 2012, 218(19): 9997 – 10006.
[6] Sun J, Liu G P, Chen J, et al. Improved delay-range-dependent stability criteria for linear systems with time-varying delays [J]. *Automatica*, 2010, 46(2): 466 – 470.
[7] Zhu X L, Wang Y. Parameter-dependent switching law for linear switched systems with time-varying delay [C]//The 9th IEEE International Conference on Control and Automation. Santiago: IEEE, 2011: 300 – 305.
[8] Khalil H K. Nonlinear systems [M]. 3rd ed. Upper Saddle River: Prentice Hall, 2002: 651 – 652.