

含裂纹压电材料的 Cell - Based 光滑有限元法

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摘 要: 裂纹缺陷限制了压电材料更广泛的应用和相关器件性能的进一步提升. 为提高求解压电材料断裂参数的精度和效率, 将复势函数法和 Cell - based 光滑有限元法引入到含裂纹平面压电材料问题中, 提出了含裂纹压电材料的 Cell - based 光滑有限元法. 以含中心裂纹压电平板的问题为例, 对不同材料、裂纹长度、网格和光滑子元数下的正则强度因子进行了讨论, 并与 FEM 作了对比, 数值算例结果表明, Cell - based 光滑有限元法改善了有限元法刚度“偏硬”的缺点, 具有高精度、高效率的优点.

关 键 词: Cell - based 光滑有限元法; 压电材料; 裂纹; 正则强度因子

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Cell-Based Smoothed Finite Element Method for Piezoelectric Materials with Cracks

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Abstract: Cracks in the piezoelectric material restricts wider application of the piezoelectric material and further improvement in the performance of related devices. To improve the accuracy and efficiency of solving the fracture parameters of piezoelectric materials, the complex potential function method and Cell-based smoothed finite element method (FEM) are introduced to fracture mechanics of piezoelectric materials, and Cell-based smoothed FEM for the piezoelectric materials with cracks was established. Normalized intensity factors of the center cracked piezoelectric with different materials, crack length, meshes and smoothing subcells were discussed and compared with FEM. Numerical example results showed that the Cell-based smoothed FEM, improving the FEM stiffness with high precision and high efficiency, is efficient numerical methods.

Key words: Cell-based smoothed finite element method; piezoelectric materials; crack; normalized intensity factor

压电断裂力学的研究成为人们关注的焦点, 始于 20 世纪 90 年代初期 Pak, Sosa 和 Suo 等^[1-3]的研究工作. Kumar 和 Singh^[4]采用奇异元对双边缺口压电试件进行了分析, 李尧臣^[5]采用有限元法计算了压电材料平面裂纹问题的强度因子和能量释放率, Fang 等^[6]采用无网格法给出了压电材料裂纹问题的力电场, 有限元法应用最为广泛, 但采用位移有限元法理论得到的位移解偏小, 为提高计算精度, Liu 等^[7]提出光滑有限元法.

光滑有限元法是将光滑应变技术引入有限元的一种方法, 目前有多种光滑有限元法^[8-9], 其中, Cell - based 光滑有限元法具有形函数选取简单、网格质量要求低、计算精度高等优点, 现已应用于压电材料领域^[10], 并取得了良好的计算结果, 但对压电体含裂纹问题还未涉及.

本文将复势函数法和 Cell - based 光滑有限元法引入到含裂纹平面压电材料问题中, 建立了含裂纹压电材料的 Cell - based 光滑有限元法, 求解了含裂纹压电材料的正则强度因子.

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1 Cell-based 光滑有限元法

单元内任一点的广义位移 \mathbf{u} 和广义电势 ϕ 可表示为

$$\mathbf{u} = \sum_{i=1}^{n_p} N_u^i q_i = N_u \mathbf{q}, \quad (1)$$

$$\phi = \sum_{i=1}^{n_p} N_\phi^i \varphi_i = N_\phi \boldsymbol{\varphi}. \quad (2)$$

式中: n_p 为单元的个数; N_u, N_ϕ 为 Cell-based 光滑有限元形函数矩阵; \mathbf{q} 和 $\boldsymbol{\varphi}$ 分别为节点位移列阵和节点电势列阵.

在光滑单元 Ω^k 内任意点 \mathbf{x}^k 处, 光滑应变 $\bar{\boldsymbol{\varepsilon}}(\mathbf{x}^k)$ 和光滑电场强度 $\bar{\mathbf{E}}(\mathbf{x}^k)$ 为

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}^k) = \int_{\Omega^k} \boldsymbol{\varepsilon}(\mathbf{x}) \Phi(\mathbf{x} - \mathbf{x}^k) d\Omega, \quad (3)$$

$$\bar{\mathbf{E}}(\mathbf{x}^k) = \int_{\Omega^k} \mathbf{E}(\mathbf{x}) \Phi(\mathbf{x} - \mathbf{x}^k) d\Omega. \quad (4)$$

式中, Φ 为光滑函数, 取

$$\Phi(\mathbf{x} - \mathbf{x}^k) = \begin{cases} 1/A^k, & \mathbf{x} \in \Omega^k; \\ 0, & \mathbf{x} \notin \Omega^k. \end{cases} \quad (5)$$

式中, A^k 为第 k 光滑元的面积, $A^k = \int_{\Omega^k} d\Omega$.

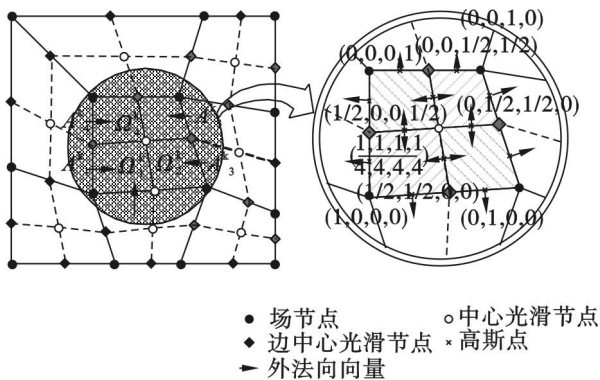


图 1 光滑子元及节点的形函数值

Fig. 1 Smoothing subcells and the values of shape functions at nodes

将求解域 Ω 划分为 n_p 个四节点单元, 节点个数为 N_d , $\Omega = \bigcup_{k=1}^{n_p} \Omega^k$, $\Omega^i \cap \Omega^j = \emptyset$, $i \neq j$, \emptyset 为空集, 再将 $\Omega_{n_c}^k$ 划分为 n_c 个光滑子元, 图 1 为四节点单元划分为 4 个光滑子元分布情况及形函数值.

将式(3)和式(4)改写为

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}^k) = \sum_{i=1}^{n_c} \bar{\mathbf{B}}_u^i(\mathbf{x}^k) \mathbf{q}_i, \quad (6)$$

$$\bar{\mathbf{E}}(\mathbf{x}^k) = \sum_{i=1}^{n_c} \bar{\mathbf{B}}_\phi^i(\mathbf{x}^k) \varphi_i. \quad (7)$$

式中, n_c 为光滑子元个数.

$$\bar{\mathbf{B}}_u^i(\mathbf{x}^k) = \frac{1}{A^k} \sum_{b=1}^{n_b} \begin{pmatrix} N_{ui}(\mathbf{x}_b^G) \mathbf{n}_x^k & 0 \\ 0 & N_{ui}(\mathbf{x}_b^G) \mathbf{n}_z^k \\ N_{ui}(\mathbf{x}_b^G) \mathbf{n}_z^k & N_{ui}(\mathbf{x}_b^G) \mathbf{n}_x^k \end{pmatrix} \mathbf{l}_b^k, \quad (8)$$

$$\bar{\mathbf{B}}_\phi^i(\mathbf{x}^k) = \frac{1}{A^k} \sum_{b=1}^{n_b} \begin{pmatrix} N_{\phi i}(\mathbf{x}_b^G) \mathbf{n}_x^k \\ N_{\phi i}(\mathbf{x}_b^G) \mathbf{n}_z^k \end{pmatrix} \mathbf{l}_b^k. \quad (9)$$

式中: N_{ui} 为 N_u 的一个元素; $N_{\phi i}$ 为 N_ϕ 的一个元素; \mathbf{x}_b^G 和 \mathbf{l}_b^k 分别为光滑边界 Γ_{cb} 的中点(高斯点)和长度; n_b 为每个光滑单元的边界总数; \mathbf{n}_x^k 和 \mathbf{n}_z^k 分别为 x 和 z 方向外法向量.

单元刚度矩阵 \mathbf{K}^k 可表示为

$$\mathbf{K}^k = \begin{bmatrix} \mathbf{K}_{uu}^k & \mathbf{K}_{u\phi}^k \\ \mathbf{K}_{u\phi}^{kT} & \mathbf{K}_{\phi\phi}^k \end{bmatrix}. \quad (10)$$

式中:

$$\mathbf{K}_{uu}^k = \sum_{i=1}^{n_c} \bar{\mathbf{B}}_u^{iT} \mathbf{C}_E \bar{\mathbf{B}}_u^i A^k, \quad (11)$$

$$\mathbf{K}_{u\phi}^k = \sum_{i=1}^{n_c} \bar{\mathbf{B}}_u^{iT} \mathbf{e}^T \bar{\mathbf{B}}_\phi^i A^k, \quad (12)$$

$$\mathbf{K}_{\phi\phi}^k = - \sum_{i=1}^{n_c} \bar{\mathbf{B}}_\phi^{iT} \mathbf{g}^T \bar{\mathbf{B}}_\phi^i A^k. \quad (13)$$

式中: \mathbf{C}_E 为弹性常数矩阵; \mathbf{e} 为压电常数矩阵; \mathbf{g} 为介电常数矩阵.

平衡方程为

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\phi} \\ \mathbf{K}_{u\phi}^T & \mathbf{K}_{\phi\phi} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\varphi} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{Q} \end{bmatrix}. \quad (14)$$

式中: $\mathbf{K}_{uu} = \sum_{n_p} \mathbf{K}_{uu}^k$; $\mathbf{K}_{u\phi} = \sum_{n_p} \mathbf{K}_{u\phi}^k$; $\mathbf{K}_{\phi\phi} = \sum_{n_p} \mathbf{K}_{\phi\phi}^k$; \mathbf{F} 为力向量; \mathbf{Q} 为电荷向量,

$$\mathbf{F} = \int_{\Omega} \mathbf{N}^T \mathbf{t} d\Omega - \int_{\Gamma} \mathbf{N}^T \mathbf{f} d\Gamma, \quad (15)$$

$$\mathbf{Q} = \int_{\Gamma_q} \mathbf{N}_\phi^T \mathbf{p} d\Gamma. \quad (16)$$

式中: \mathbf{t} 为定面力; \mathbf{p} 为面电荷密度; \mathbf{f} 为体力.

2 压电断裂力学

无限大均匀压电体含 $2a$ 长度裂纹, 无穷远处作用的力电载荷为 $\mathbf{t}_3^\infty = [\sigma_{31}^\infty \quad \sigma_{33}^\infty \quad D_3^\infty]^T$, 裂纹面上应力和电位移为零, 采用复变函数法求解压电材料的机电耦合裂纹尖端场^[5], 引入复变应力函数 $U(z)$ 和电位移函数 $\varphi(z)$, 复势的解为

$$\varphi_k(z_k) = (B_k + iB_k^*) z_k + \frac{1}{a} \sum_{j=1}^3 A_{kj} l_j [z_k - \sqrt{z_k^2 - a^2}], \quad (17)$$

$$\varphi_k'(z_k) = (B_k + iB_k^*) + \frac{1}{a} \sum_{j=1}^3 \Lambda_{kj} l_j \left[1 - \frac{z_k}{\sqrt{z_k^2 - a^2}} \right]. \quad (18)$$

式中: $l_1 = -a\sigma_{33}^\infty/2$; $l_2 = a\sigma_{31}^\infty/2$; $l_3 = aD_3^\infty/2$; B_k 和 B_k^* 为实数, 由远场的载荷条件决定; z_k 为复数形式的参数 ($k=1, 2, 3$),

$$[\Lambda_{kj}] = \frac{1}{\Delta} \begin{bmatrix} \mu_2 \lambda_3 - \mu_3 \lambda_2 & \lambda_2 - \lambda_3 & \mu_3 - \mu_2 \\ \mu_3 \lambda_1 - \mu_1 \lambda_3 & \lambda_3 - \lambda_1 & \mu_1 - \mu_3 \\ \mu_1 \lambda_2 - \mu_2 \lambda_1 & \lambda_1 - \lambda_2 & \mu_2 - \mu_1 \end{bmatrix}. \quad (19)$$

式中, μ_i 为该问题特征方程的根.

$$\Delta = (\lambda_2 - \lambda_3)\mu_1 + (\lambda_3 - \lambda_1)\mu_2 + (\lambda_1 - \lambda_2)\mu_3, \quad (20)$$

$$\lambda_k = -b(\mu_k)/\delta(\mu_k), \delta(\mu_k) \neq 0, \quad (21)$$

$$b(\mu_k) = (b_{31} + b_{15})\mu_k^2 + b_{33}, \quad (22)$$

$$\delta(\mu_k) = \delta_{11}\mu_k^2 + \delta_{33}. \quad (23)$$

考察裂尖前方的材料行为, 令 $x_3 = 0$, $z_k = x_1 = a + \Delta a$, $\Delta a \rightarrow 0$. 代入式(17)和式(18)得

$$\varphi_k(z_k) = Ma + \frac{1}{a} \sum_{j=1}^3 \Lambda_{kj} l_j [a + \Delta a - \sqrt{2a\Delta a}], \quad (24)$$

$$\varphi_k'(z_k) = M + \frac{1}{a} \sum_{j=1}^3 \Lambda_{kj} l_j \left[1 - \frac{a + \Delta a}{\sqrt{2a\Delta a}} \right]. \quad (25)$$

式中:

$$M = B_k + iB_k^*, \quad (26)$$

定义

$$K = \sum_{k=1}^3 K_k = -2 \sqrt{\frac{\pi}{a}} \sum_{k=1}^3 \sum_{i=1}^3 \Lambda_{ki} l_i, \quad (27)$$

则

$$K_I = \text{Re} \sum_{k=1}^3 K_k, \quad (28)$$

$$K_{II} = -\text{Re} \sum_{k=1}^3 \mu_k K_k, \quad (29)$$

$$K_D = -\text{Re} \sum_{k=1}^3 \lambda_k K_k. \quad (30)$$

当压电材料受远场载荷作用时, 应力强度因子和电位移强度因子为

$$K_I = \sigma_{33}^\infty \sqrt{\pi a}, \quad (31)$$

$$K_{II} = \sigma_{31}^\infty \sqrt{\pi a}, \quad (32)$$

$$K_D = D_{33}^\infty \sqrt{\pi a}. \quad (33)$$

裂纹尖端前方的位移和电势为

$$u_1 = 2a \text{Re} \sum_{k=1}^3 p_k M - N \text{Re} \sum_{k=1}^3 p_k K_k + \omega x_3 + u_0, \quad (34)$$

$$u_3 = 2a \text{Re} \sum_{k=1}^3 q_k M - N \text{Re} \sum_{k=1}^3 q_k K_k - \omega x_1 + v_0, \quad (35)$$

$$\phi_1 = -2a \text{Re} \sum_{k=1}^3 C_k M + N \text{Re} \sum_{k=1}^3 C_k K_k + \phi_0. \quad (36)$$

式中:

$$N = \frac{a + \Delta a - \sqrt{2a\Delta a}}{\sqrt{\pi a}}, \quad (37)$$

$$p_k = a_{11}\mu_k^2 + a_{13} - b_{31}\lambda_k, \quad (38)$$

$$q_k = a_{13}\mu_k^2 + a_{33} - b_{33}\lambda_k)/\mu_k, \quad (39)$$

$$C_k = (b_{15} + \delta_{11}\lambda_k)\mu_k. \quad (40)$$

式中: ω , u_0 和 v_0 为刚体位移, ϕ_0 为参照电势.

裂尖位移和电势为

$$u_1^{(0)} = 2a \text{Re} \sum_{k=1}^3 p_k M - \sqrt{\frac{a}{\pi}} \text{Re} \sum_{k=1}^3 p_k K_k + \omega x_3 + u_0, \quad (41)$$

$$u_3^{(0)} = 2a \text{Re} \sum_{k=1}^3 q_k M - \sqrt{\frac{a}{\pi}} \text{Re} \sum_{k=1}^3 q_k K_k - \omega x_1 + v_0, \quad (42)$$

$$\phi_1^{(0)} = -2a \text{Re} \sum_{k=1}^3 C_k M + \sqrt{\frac{a}{\pi}} \text{Re} \sum_{k=1}^3 C_k K_k + \phi_0. \quad (43)$$

得

$$\lim_{\Delta a \rightarrow 0} \left[\frac{u_1 - u_1^{(0)}}{\sqrt{\Delta a}} \right] = \sqrt{\frac{2}{\pi}} \text{Re} \sum_{k=1}^3 p_k K_k, \quad (44)$$

$$\lim_{\Delta a \rightarrow 0} \left[\frac{u_3 - u_3^{(0)}}{\sqrt{\Delta a}} \right] = \sqrt{\frac{2}{\pi}} \text{Re} \sum_{k=1}^3 q_k K_k, \quad (45)$$

$$\lim_{\Delta a \rightarrow 0} \left[\frac{\phi - \phi_1^{(0)}}{\sqrt{\Delta a}} \right] = -\sqrt{\frac{2}{\pi}} \text{Re} \sum_{k=1}^3 C_k K_k. \quad (46)$$

因此, 采用 Cell-based 光滑有限元法求得节点位移和节点电势的数值解代入式(44), 式(45)和式(46)便可求解强度因子.

3 数值算例

在力电耦合载荷作用下, 含中心裂纹压电体模型如图2所示, 边长 $2l = 40 \text{ cm}$, 裂纹长度 $2a$,

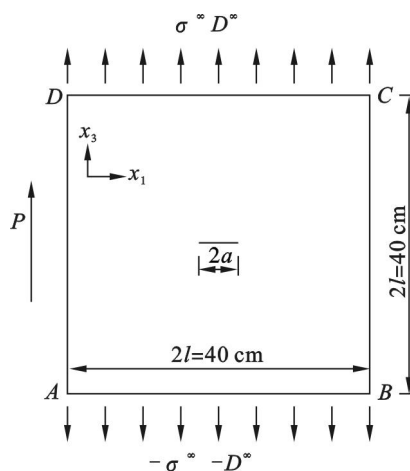


图2 含中心裂纹压电体模型

Fig. 2 Piezoelectric model with a centre crack

P 为极化方向,受到无穷远处的单向均匀拉伸 $\sigma^\infty = 0.1 \text{ MPa}$ 和电位移 $D^\infty = 7.5 \times 10^{-5} \text{ C/m}^2$,采用 PZT-4 和 P-7 压电材料进行数值模拟,材料参数如表 1 所示.采用 Cell-based 光滑有限元计算 u_3 和 ϕ ,再外推到 $\Delta a = 0$ 处,联立方程组,便可求得强度因子.

表 1 材料常数
Table 1 Material constant

材料	弹性常数 $\times 10^{-10}/(\text{N}\cdot\text{m}^{-2})$					压电常数/ $(\text{C}\cdot\text{m}^{-2})$			介电常数 $\times 10^{-10}/(\text{C}\cdot\text{V}\cdot\text{m}^{-1})$	
	c_{11}	c_{12}	c_{13}	c_{44}	c_{55}	e_{31}	e_{33}	e_{15}	d_{11}	d_{33}
PZT-4	13.9	7.78	7.43	2.56	11.3	-6.98	13.84	13.44	60.0	54.7
P-7	13.0	8.3	8.3	2.5	11.9	-10.3	14.7	13.5	171.0	186.0

由于结构对称,取试件 1/4 进行计算,对左端所有节点的 x_1 方向位移进行约束,底部的裂尖以右的所有节点进行 x_3 方向约束和电势约束(即底部电势为零),同时在其顶部施加相应的应力和电位移.对裂纹长度 $2a = 2 \text{ cm}$ 情况下,采用图 3~图 5 所示的开裂前三种网格划分模式(I 为裂尖圆形区域加密,II 为裂尖矩形区域加密,III 为裂尖均匀加密),计算时采用 4 个光滑子元,并与 FEM 作比较,给出了正则强度因子的计算结果,如表 2 所示.

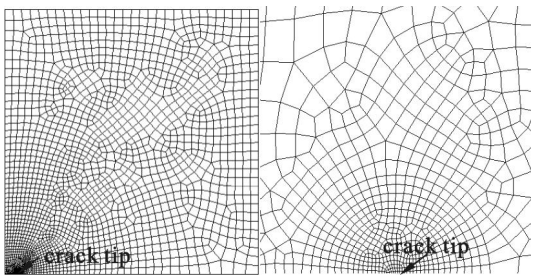


图 3 裂尖圆形区域加密 (Nodes:2508, Elements:2407)
Fig. 3 Round refinement at crack tip (Nodes:2508, Elements:2407)

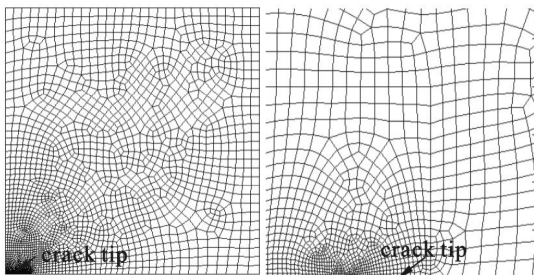


图 4 裂尖矩形区域加密 (Nodes:2754,Elements:2643)
Fig. 4 Rectangular refinement at crack tip (Nodes:2754,Elements:2643)

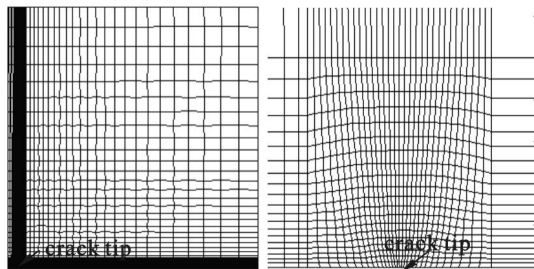


图 5 裂尖均匀加密 (Nodes:2745,Elements:2640)
Fig. 5 Uniform refinement at crack tip (Nodes:2745,Elements:2640)

表 2 不同网格划分下正则强度因子
Table 2 Normalized intensity factor under different mesh dividing methods

强度因子	I				II				III			
	FEM		SFEM		FEM		SFEM		FEM		SFEM	
	PZT-4	P-7	PZT-4	P-7	PZT-4	P-7	PZT-4	P-7	PZT-4	P-7	PZT-4	P-7
$K_I/\sigma_{33}^\infty \sqrt{\pi a}$	0.964	0.977	0.972	0.988	0.950	0.950	0.956	0.961	0.963	0.940	0.973	0.955
$K_D/D_{33}^\infty \sqrt{\pi a}$	0.959	0.985	0.967	0.994	0.937	0.957	0.948	0.968	0.918	0.925	0.938	0.944

由表 2 可以看出,SFEM 和 FEM 在 3 种模型二种材料下均得到了精度较高的正则强度因子,SFEM 计算精度高于 FEM,在裂尖处采用圆形区域加密精度最高,后面计算均采用该种局部加密形式;同时还对 I 模型下 SFEM 和 FEM 计算效率作了研究,在 CPU: Intel(R) Core(TM) i5-3470 3.20 GHz, RAM: 8 GB 情况下, SFEM 耗时

30.975 s, FEM 耗时 32.058 s, SFEM 效率有所提高,但不明显.

表 3 给出了不同裂纹长度,每个 4 节点网格均采用 4 个光滑子元时 SFEM 和 FEM 所得正则强度因子,从结果可以看出 SFEM 精度高于 FEM.

表 3 不同裂纹长度下正则强度因子
Table 3 Normalized intensity factor under different crack lengths

$\frac{a}{\text{cm}}$	强度因子	PZT – 4		P – 7	
		FEM	SFEM	FEM	SFEM
1	$K_I/\sigma_{33}^\infty \sqrt{\pi a}$	0.964	0.972	0.977	0.988
	$K_D/D_{33}^\infty \sqrt{\pi a}$	0.959	0.967	0.985	0.994
1.5	$K_I/\sigma_{33}^\infty \sqrt{\pi a}$	0.932	0.947	0.924	0.941
	$K_D/D_{33}^\infty \sqrt{\pi a}$	0.940	0.955	0.947	0.963
2	$K_I/\sigma_{33}^\infty \sqrt{\pi a}$	0.980	0.987	0.971	0.983
	$K_D/D_{33}^\infty \sqrt{\pi a}$	0.969	0.978	0.978	0.987
2.5	$K_I/\sigma_{33}^\infty \sqrt{\pi a}$	0.987	0.993	0.979	0.991
	$K_D/D_{33}^\infty \sqrt{\pi a}$	0.974	0.983	0.983	0.993
3	$K_I/\sigma_{33}^\infty \sqrt{\pi a}$	0.990	0.994	0.996	1.003
	$K_D/D_{33}^\infty \sqrt{\pi a}$	0.991	0.998	0.981	0.989

表 4 给出了裂纹长度 $2a = 6 \text{ cm}$, 材料为 P – 7, 光滑子元采用 1, 2, 3, 4, 8, 16 时, SFEM 所得正则强度因子, 从结果可以看出光滑子元为 2 时, SFEM 精度就具有很高精度, 从一致性方面验证了 SFEM 的正确性和可靠性.

表 4 不同光滑子元个数下正则强度因子
Table 4 Normalized intensity factor under different smoothing subcells

强度因子	FEM	SFEM					
		1	2	3	4	8	16
$K_I/\sigma_{33}^\infty \sqrt{\pi a}$	0.996	0.957	1.009	1.011	1.003	1.003	0.998
$K_D/D_{33}^\infty \sqrt{\pi a}$	0.981	0.921	0.998	0.993	0.989	0.985	0.986

4 结 论

将复势函数法和光滑有限元法引入到含裂纹平面压电材料问题中, 提出了含裂纹压电材料的 Cell – based 光滑有限元法, 并与 FEM 的精度和效率做了对比. 采用 SFEM 的精度高于 FEM, 裂尖圆形区域加密比裂尖矩形区域加密和均匀加密精度高, 且具有很好的一致性.

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